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CONVERGENCE OF U-STATISTICS INDEXED BY A RANDOM WALK TO STOCHASTIC INTEGRALS OF A LÉVY SHEET

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Abstract. A U-statistic indexed by a $\mathbb{Z}^{d_0}$-random walk $(S_n)_n$ is a process $U_n := \sum_{i,j=1}^n h(\xi_{S_i}, \xi_{S_j})$ where h is some real-valued function and $(\xi_k)_k$ is a sequence of iid random variables, which are independent of the walk. Concerning the walk, we assume either that it is transient or that its increments are in the normal domain of attraction of a strictly stable distribution of exponent $\alpha \in [d_0, 2)$. We further assume that the distribution of $h(\xi_1, \xi_2)$ belongs to the normal domain of attraction of a strictly stable distribution of exponent $\beta \in (0, 2)$. For a suitable renormalization $(a_n)_n$ we establish the convergence in distribution of the sequence of processes $(U_{\lfloor nt \rfloor}/a_n)_t; n \in \mathbb{N}$ to some suitable observable of a Lévy sheet $(Z_{s,t})_{s,t}$. The limit process is the diagonal process $(Z_{t,t})_t$ when the walk is transient or when $\alpha = d_0$. When $\alpha > d_0 = 1$ the limit process is some stochastic integral with respect to Z.

1. Introduction

Given a random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}^{d_0}$ and a sequence of independent identically distributed (iid) real random variables $(\xi_k)_{k\in\mathbb{Z}^{d_0}}$, independent one from each other, one can consider the random walk in random scenery $\mathcal{S}_n := \sum_{k=1}^n \xi_{S_k}$. In particular one is interested in the limit behavior of the sequence of renormalized processes $(\nu_n^{-1}\mathcal{S}_{[nt]})_{t\geq 0}; n \in \mathbb{N}$. In this context the following assumptions are usually made:

(A) either S_n is transient or there exists some $\alpha \in [d_0, 2]$ such that $n^{-\frac{1}{\alpha}} S_n; n \in \mathbb{N}$ converges in distribution to a random variable;

(B) $n^{-\frac{1}{\beta}} \sum_{k=1}^n \xi_k; n \in \mathbb{N}$ converges in distribution to a random variable for some $\beta \in (0, 2]$.

Note that in the case $\alpha > d_0 = 1$ the assumption (A) implies that the sequence of stochastic processes $(n^{-\frac{1}{\alpha}} S_{\lfloor nt \rfloor})_{t>0}; n \in \mathbb{N}$ converges in distribution to some α-stable Lévy process $(Y_t)_{t>0}$ which admits a local time $(\mathcal{L}_t(x),\ t \geq 0,\ x \in \mathbb{R})$. Similarly, assumption (B) implies that $(n^{-\frac{1}{\beta}} \sum_{k=1}^{\lfloor nt \rfloor} \xi_k)_{t>0}; n \in \mathbb{N}$ converges in distribution to some β-stable process $(\mathcal{Z}_t)_{t>0}$.[1] Subsequently we will use $(\mathcal{Z}_{-t})_{t>0}$ to denote an independent copy of $(\mathcal{Z}_t)_{t>0}$.

Random walks in random scenery have been studied by many authors since the early works of Borodin [4, 5] and Kesten and Spitzer [17]. In particular, [3, 11, 7] complete the study of the limit in distribution of random walks in random scenery. The asymptotic behavior of the sequence $(\nu_n^{-1}\mathcal{S}_{\lfloor nt \rfloor})_{t>0}; n \in \mathbb{N}$ is summarized in the following table (where d_1 and d_2 are explicit constants depending on (S_n) and on β):

Cases	normalization	Limit process	Space of convergence in distribution
transient	$\nu_n := n^{\frac{1}{\beta}}$	$(d_1\mathcal{Z}_t)_t$	finite distributions if $\beta \neq 1$:Skorokhod space with M_1-metric
$\alpha = d_0$	$\nu_n := n^{\frac{1}{\beta}}(\log n)^{1-\frac{1}{\beta}}$	$(d_2\mathcal{Z}_t)_t$	finite distributions if $\beta \neq 1$:Skorokhod space with M_1-metric
$\alpha > d_0$	$\nu_n := n^{1-\frac{1}{\alpha}+\frac{1}{\alpha\beta}}$	$(\Delta_t := \int_{\mathbb{R}^*} \mathcal{L}_t(x)\, d\mathcal{Z}_x)_t$	Skorokhod space with J_1-metric

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[1] to simplify notations, for every $k \in \mathbb{Z}$, we write ξ_k for $\xi_{(k,\dots,k)}$

In this paper we want to do a similar investigation for U-statistics indexed by a random walk. To introduce the objects let E be some measurable set and $(\xi_k)_{k\in\mathbb{Z}^{d_0}}$ an iid sequence of E-valued random variables. Often we might abbreviate this family of random variables by ξ and call it the scenery. Moreover, let $(S_n)_{n\geq 1}$ be as above a random walk on $\mathbb{Z}^{d_0}$, which is independent of the scenery ξ. We will also use the short notation S for the random walk. For some measurable function $h : E^2 \to \mathbb{R}$, we consider the U-statistic indexed by S defined through

$$U_n := \sum_{i,j=1}^{n} h(\xi_{S_i}, \xi_{S_j}).$$

We are interested in results of distributional convergence for $(U_n)_n$ (after some suitable normalization) under the assumption that the distribution of $h(\xi_1, \xi_2)$ is in the normal domain of attraction of a β-stable distribution. Let us assume without loss of generality that h is symmetric.

If $\beta > 1$ we can introduce $\vartheta_k := \mathbb{E}[h(\xi_0, \xi_k)|\xi_0]$. Two different situations can occur. We will say that the kernel is degenerate if $\vartheta_1 = 0$ almost surely. Otherwise, we will say that the kernel is non-degenerate.

The case when when $h(\xi_1, \xi_2)$ is square integrable and centered (which implies $\beta = 2$) has been fully studied by Guillotin-Plantard and her co-authors. In this case only two kind of behaviors can occur:

(a) the kernel is non-degenerate, then one can use Hoeffding decomposition to show that U_n behaves essentially as $\sum_{i,j=1}^n(\vartheta_{S_i} + \vartheta_{S_j}) = 2n\sum_{i=1}^n \vartheta_{S_i}$.
(b) the kernel is degenerate, then Hilbert-Schmidt theory can be used to represent the kernel as $h(x, y) = \sum_p \lambda_p\phi_p(x)\phi_p(y)$ and to show that U_n behaves as $\sum_p \lambda_p(\sum_{i=1}^n \phi_p(S_i))^2$.

This has been proved by Cabus and Guillotin-Plantard in [6] for random walks in $\mathbb{Z}^{d_0}$ with $d_0 \geq 2$ and by Guillotin-Plantard and Ladret in [15] for random walks in $\mathbb{Z}$.

Note that the situation treated in [6] splits into the case $d_0 > 2$, where the walk is transient, and the singular case $d_0 = 2$, where the random walk is null recurrent. However, in this last case the limit process $(Y_t)_{t\geq 0}$ does not have local time. In contrast to this the assumptions made in [15] correspond to some null recurrent random walk with existing local time for $(Y_t)_{t\geq 0}$; i.e.: $\alpha > d_0 = 1$.

The special form of the representations given in (a) and (b) implies that for $\beta = 2$, the study of $(U_n)_n$ can be reduced to the study of some suitable random walk in random scenery (either $\sum_{i=1}^n \vartheta_{S_i}$ or $\sum_{i=1}^n \phi_p(S_i)$). Thus the limits can be expressed in terms of processes which already occurred in the random scenery situation.

In the transient case or if $d_0 = 2$ the limit process turns out to be Brownian motion $(B_t)_{t\geq 0}$ when the kernel is non-degenerate. In the degenerate situation the limit has the representation $\sum_p \lambda_p(B_t^{(p)})^2$, where $(B_t^{(p)})_{t\geq 0}; p \in \mathbb{N}$ is a sequence of independent Brownian motions (see [6]).

If on the other hand $\alpha > d_0 = 1$, then in the non-degenerate situation the limit is the usual process $\Delta_t := \int_{\mathbb{R}^*} \mathcal{L}_t(x)\, dB_x$, where $(B_x)_{x>0}$ and $(B_{-x})_{x>0}$ are independent one-dimensional Brownian motions. In the degenerate case the limit takes the form $\sum_p \lambda_p\big(\int_{\mathbb{R}^*} \mathcal{L}_t(x)\, dB_x^{(p)}\big)^2$, where the pairs $(B_x^{(p)})_{x>0}$, $(B_{-x}^{(p)})_{x>0}$ form a sequence of independent copies of the pair $(B_x)_{x>0}$, $(B_{-x})_{x>0}$ (see [15]).

Let us further mention that (a) includes the case where $h(x, y) = g(x) + g(y)$ and that (b) includes the case when $h(x, y) = g(x)g(y)$. Here $g : E \to \mathbb{R}$ is a measurable function such that $g(\xi_1)$ is square integrable and centered.

When $1 < \beta < 2$, a similar behavior can occur in the non-degenerate case. For instance, in [14], we use Hoeffding decomposition to prove the following:

(a') If $1 < \beta \leq 2$ and if the distribution of ϑ_1 is in the normal domain of attraction of a β-stable distribution then U_n behaves as $2n\sum_{i=1}^n \vartheta_{S_i}$.

This holds for example if $h(x,y) = g(x) + g(y)$. The limit then turns out to be β-stable Lévy process $(Z_t)_{t\geq 0}$ when the walk is transient or when $\alpha = d_0$. However, when $\alpha > d_0$ the limit has the representation $\Delta_t := \int_{\mathbb{R}^*} \mathcal{L}_t(x)\, dZ_x$, where $(Z_x)_{x>0}$ and $(Z_{-x})_{x>0}$ are independent one-dimensional β-stable Lévy-motions (see [14]).

On the other hand in the degenerate case, when $\vartheta_1 = 0$, different limits than those described in (b) can arise when $0 < \beta < 2$. This is the purpose of the present paper. The limit we obtain is the diagonal process $(Z_{(t,t)})_{t\geq 0}$ of a Lévy sheet $(Z_{t,s})_{t,s\geq 0}$, when the walk is transient or when $\alpha = d_0$, and a stochastic integral $\int_{\mathbb{R}^2} L_t(x)L_t(y)dZ_{x,y}$ with respect to four independent copies of the Lévy sheet introduced above, when $\alpha > d_0$. These limits can be understood as two-dimensional analogues of the known limits for random walk in random scenery found by Kesten and Spitzer (see [17]).

To be more precise, let us keep assumption **(A)** but replace **(B)** on $(\xi_k)_k$ by the following assumption on $(h(\xi_k, \xi_\ell))_{k,\ell}$:

(B') $(n^{-\frac{1}{\beta}} \sum_{k=1}^n h(\xi_{2k}, \xi_{2k+1}))_n$ converges in distribution to a random variable with $\beta \in (0,2)$.

This implies that if $(h_{i,j})_{i,j}$ is a sequence of iid random variables with the same distribution as $h(\xi_1, \xi_2)$, then the sequence of stochastic processes $(n^{-\frac{2}{\beta}} \sum_{k=1}^{\lfloor nt\rfloor} \sum_{\ell=1}^{\lfloor ns\rfloor} h_{i,j})_{t>0}; n \in \mathbb{N}$ converges in law to some β-stable Lévy sheet $(Z_{s,t})_{s,t>0}$ (which we extend on $\mathbb{R}^2$).

In the present paper, under assumption (B') and some additional assumptions, we prove limit theorems for the U-statistic which are summarized in the following table:

Cases	normalization	Limit process	Space of convergence in distribution
transient	$\nu_n^2 = n^{\frac{2}{\beta}}$	$(d_1^2 Z_{t,t})_t$	finite distribution
$\alpha = d_0$	$\nu_n^2 = n^{\frac{2}{\beta}} (\log n)^{2-\frac{2}{\beta}}$	$(d_2^2 Z_{t,t})_t$	finite distribution
$\alpha > d_0$	$\nu_n^2 = n^{2-\frac{2}{\alpha}+\frac{2}{\alpha\beta}}$	$(\int_{\mathbb{R}^2} \mathcal{L}_t(x)\mathcal{L}_t(y)\, dZ_{x,y})_t$	Skorokhod space with J_1-metric

The present paper is organized as follows. The assumptions and main results are stated in Section 2. We give some examples which satisfy our assumptions in Section 3. We prove our results concerning convergence of finite distribution in Section 4. In the spirit of [10], our proof relies on the convergence of a suitably defined point process to a Poisson point process which is established by the use of Kallenberg theorem. In Section 5, we prove the tightness for the J_1-metric when $\alpha > d_0$. We complete our article with some facts on the β-stable Lévy sheet Z in Appendix A. In particular a construction of stochastic integrals with respect to Z is given.

2. Main results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a suitable probability space and let $S = (S_n)_{n\geq 0}$ be a $\mathbb{Z}^{d_0}$-valued random walk on $(\Omega, \mathcal{F}, \mathbb{P})$ with $S_0 = 0$ such that one of the following conditions holds:

- the random walk $(S_n)_{n\geq 0}$ is transient,
- the random walk $(S_n)_{n\geq 0}$ is recurrent and there exists $\alpha \in [d_0, 2]$ such that $(n^{-\frac{1}{\alpha}} S_n)_{n\geq 1}$ converges in distribution to a random variable Y. In this case we further assume that $\forall x \in \mathbb{Z}^{d_0}, \exists n \in \mathbb{N} : \mathbb{P}(S_n = x) > 0$.

Recall that, in the second case, $(n^{-\frac{1}{\alpha}} S_{\lfloor nt\rfloor})_{t>0}; n \in \mathbb{N}$ converges in distribution to an α-stable process $(Y_t)_{t>0}$ such that Y_1 has the same law as Y.

In order to get a uniform notation for the different situations, we define α_0 to be a number, which is one when the random walk is transient, and which takes the value $\frac{\alpha}{d_0}$ in the recurrent case.

Let $\xi = (\xi_\ell)_{\ell\in\mathbb{Z}^{d_0}}$ be a family of iid random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in some measurable space E. We assume that the two families S and ξ are independent. Let $h : E \times E \to \mathbb{R}$ be a measurable function. We are interested in the properties of the U-statistics process $U_n := \sum_{i,j=1}^n h(\xi_{S_i}, \xi_{S_j})$. In this work, we assume moreover that the following properties are satisfied.

Assumption 1. Let $\beta \in (0, 2)$.

(i) for every $x \in E$, $h(x, x) = 0$;

(ii) h symmetric (i.e. $h(x, y) = h(y, x)$ for every $x, y \in E$);

(iii) there exist $c_0, c_1 \in [0, +\infty)$ with $c_0 + c_1 > 0$ such that

$$\forall z > 0, \quad \mathbb{P}(h(\xi_1, \xi_2) \geq z) = z^{-\beta} L_0(z), \quad \text{with} \quad \lim_{z \to +\infty} L_0(z) = c_0; \tag{1}$$

and

$$\forall z > 0, \quad \mathbb{P}(h(\xi_1, \xi_2) \leq -z) = z^{-\beta} L_1(z), \quad \text{with} \quad \lim_{z \to +\infty} L_1(z) = c_1; \tag{2}$$

(iv) there exist $C_0 > 0$ and $\gamma > \frac{3\beta}{4}$ such that

$$\forall z, z' \in (0, +\infty), \quad \mathbb{P}\Big(|h(\xi_1, \xi_2)| \geq z \text{ and } |h(\xi_1, \xi_3)| \geq z'\Big) \leq C_0 \big(\max(1, z) \max(1, z')\big)^{-\gamma}; \tag{3}$$

(v) If $\beta > 1$, then $\mathbb{E}[h(\xi_1, \xi_2)] = 0$;

(vi) If $\beta \geq 4/3$, there exists $C_0' > 0$ and $\theta' > \frac{3\beta}{4} - 1$ such that

$$\forall M, M' \in (0, +\infty), \quad |\mathbb{E}\left[\mathbf{h}_M(\xi_1, \xi_2)\mathbf{h}_{M'}(\xi_1, \xi_3)\right]| \leq C_0'(MM')^{-\theta'},$$

where $\mathbf{h}_M(x, y) := h(x, y)\mathbf{1}_{\{|h(x,y)| \leq M\}} + \frac{\beta}{\beta - 1}(c_0 - c_1)M^{1-\beta}$.

(vii) If $\beta = 1$, then $c_0 = c_1$ and $\lim_{M \to +\infty} \mathbb{E}[h(\xi_1, \xi_2)\mathbf{1}_{\{|h(\xi_1,\xi_2)| \leq M\}}] = 0$.

Some examples satisfying the above assumptions are presented in the next section.

Remark 2. The following comments on the different points in Assumptions 1 might be of some help:

- Item (i) can be relaxed as will be proved in Proposition 7 below.
- Item (ii) is not restrictive since one can always replace $h(z, z')$ by $(h(z, z') + h(z', z))/2$ without changing the sequence $(U_n)_n$.
- Note that Item (iv) is a condition which ensures that the tail behavior resulting from coupling of the pairs (ξ_1, ξ_2) and (ξ_1, ξ_3) does not interfere with the tail behavior of the single terms $h(\xi_1, \xi_2)$. A condition with the same spirit is condition (2.1) in [10].
- If Item (iii) holds and if for every $x \in E$ the distribution of $h(x, \xi_1)$ is symmetric, then Item (vi) and Item (vii) are also satisfied. Indeed, in this case, $c_0 = c_1$ and
$$\mathbb{E}\left[\mathbf{h}_M(\xi_1, \xi_2)\mathbf{h}_{M'}(\xi_1, \xi_3)\right] = \int_E \mathbb{E}[h(x, \xi_2)\mathbf{1}_{\{|h(x,\xi_2)| \leq M\}}]\mathbb{E}[h(x, \xi_2)\mathbf{1}_{\{|h(x,\xi_2)| \leq M'\}}]\, d\mathbb{P}_{\xi_1}(x) = 0.$$
- Note that Item (iii) and Item (v) imply that the law of $h(\xi_1, \xi_2)$ is in the domain of attraction of a β-stable law for some $\beta \in (0, 2)$.

Let $(h_{i,j})_{i,j}$ be a sequence of iid random variables with same distribution as $h(\xi_1, \xi_2)$. Observe that the Items (i), (iii), (v) and (vii) in Assumption 1 describe the classical situation, where the sequence of random fields $(n^{-\frac{2}{\beta}} \sum_{i=1}^{\lfloor nx \rfloor} \sum_{j=1}^{\lfloor ny \rfloor} h_{i,j})_{x,y>0}; n \in \mathbb{N}$ converges in law to a β-stable Lévy sheet $(\tilde{Z}_{x,y})_{x,y \geq 0}$ such that the characteristic function of $\tilde{Z}_{x,y}$ is given by $\mathbb{E}[e^{iz\tilde{Z}_{x,y}}] = \Phi_{xy(c_0+c_1), xy(c_0-c_1), \beta}(z)$, with

$$\Phi_{A,B,\beta}(z) := \exp\left(-|z|^\beta \int_0^{+\infty} \frac{\sin t}{t^\beta}\, dt \left(A - iB\,\mathrm{sgn}(z) \tan\frac{\pi\beta}{2}\right)\right) \quad \text{if } \beta \neq 1 \tag{4}$$

and

$$\Phi_{A,B,1}(z) := \exp\Big(-|z|\left(\frac{\pi}{2}A + iB\,\mathrm{sgn}(z)\log|z|\right)\Big) \tag{5}$$

(see [13, p. 568-569]). In order to construct a continuation of the Lévy sheet $\tilde{Z}$ to all of $\mathbb{R}^2$ we use four independent copies $Z^{(\varepsilon,\varepsilon')}$ (with $\varepsilon,\varepsilon' \in \{1,-1\}$) of $\tilde{Z}$ to introduce $Z_{x,y} := Z^{(\mathrm{sgn}(x),\mathrm{sgn}(y))}_{|x|,|y|}$ for all $(x,y) \in \mathbb{R}^2$. In the following we will need to integrate some continuous compactly supported function ψ with respect to Z, i.e.:

$$\int_{\mathbb{R}^2} \psi(x,y)\, dZ_{x,y}.$$

More information on Lévy sheets and on the construction of the integral can be found in Appendix A.

When $\alpha > d_0 = 1$, we assume moreover that $(Z_{x,y})_{x,y}$ is independent of the α-stable process $(Y_t)_t$.

If the random walk is transient, we write N_∞ for the total number of visits of the two sided random walk $(S_n)_{n\in\mathbb{Z}}$ to zero; i.e.: $N_\infty := \sum_{n\in\mathbb{Z}} \mathbf{1}_{\{S_n=0\}}$.

Theorem 3 (Transient case)**.** *Suppose $(S_n)_{n\geq 0}$ is transient and Assumption 1. We set $a_n := n^{\frac{2}{\beta}}$. Then the finite distributions of $((U_{\lfloor nt\rfloor}/a_n)_{t>0})_n$ converge to the finite distributions of $(K_\beta^{\frac{2}{\beta}} Z_{t,t})_{t>0}$, with $K_\beta := \mathbb{E}[N_\infty^{\beta-1}]$.*

In particular the previous theorem holds for the deterministic $\mathbb{Z}$-valued walk $S_n = n$ (for which $K_\beta = 1$). In that case our result boils down to a result on classical U-statistics which was established by Dabrowski, Dehling, Mikosch and Sharipov in [10]. We emphasize this point in the following corollary, since the link to the Lévy sheet was not mentioned in [10].

Corollary 4 (Deterministic case)**.** *Suppose Assumption 1 and set $a_n := n^{\frac{2}{\beta}}$. The finite distributions of $((\sum_{i,j=1}^{\lfloor nt\rfloor} h(\xi_i,\xi_j)/a_n)_{t>0})_n$ converge to the finite distributions of $(Z_{t,t})_{t>0}$.*

As usual Γ will stand for the Gamma function. We also write $N_n(x)$ for the occupation time of S at x up to time n, i.e.:

$$N_n(x) := \sum_{i=1}^{n} \mathbf{1}_{\{S_i=x\}}.$$

We define the maximal occupation time of S up to time n through $N_n^* := \max_x N_n(x)$ and the range of S up to time n by

$$R_n := \#\{y \in \mathbb{Z}^{d_0} \,:\, N_n(y) > 0\}.$$

We recall that, when $\alpha = d_0$, there exists $c_3 > 0$ such that

$$R_n \sim c_3 n/\log n \ a.s. \ \text{ as } \ n \to \infty. \tag{6}$$

Theorem 5 (Recurrent case without local time)**.** *Suppose $\alpha = d_0 \in \{1,2\}$ and Assumption 1. We set $a_n := n^{\frac{2}{\beta}}(\log n)^{2-\frac{2}{\beta}}$. Then the finite distributions of $((U_{\lfloor nt\rfloor}/a_n)_{t>0})_n$ converge to the finite distributions of $(K_\beta^{\frac{2}{\beta}} Z_{t,t})_{t>0}$, with $K_\beta := \Gamma(\beta+1)/c_3^{\beta-1}$ and with c_3 given by (6).*

When $\alpha > d_0$ (which implies $d_0 = 1$), we prove a result of convergence in distribution in the Skorokhod space for the J_1-metric. Recall that $\mathbf{h}_M(x,y) = h(x,y)\mathbf{1}_{\{|h(x,y)|\leq M\}} + \frac{\beta}{\beta-1}(c_0 - c_1)M^{1-\beta}$.

Theorem 6 (Recurrent case with local time)**.** *Assume $\alpha \in (1,2]$, $d_0 = 1$ and Assumption 1. We set $a_n := n^{2\delta}$ with $\delta = 1 - \frac{1}{\alpha} + \frac{1}{\alpha\beta}$. Then, for every $T > 0$, $((U_{\lfloor nt\rfloor}/a_n)_{t\in[0,T]})_n$ converges in distribution (in the Skorokhod space $D([0,T])$ endowed with the J_1 metric) to $(\int_{\mathbb{R}^2} \mathcal{L}_t(x)\mathcal{L}_t(y)\, dZ_{x,y})_{t\in[0,T]}$, where $(\mathcal{L}_t(x),\ t \geq 0,\ x \in \mathbb{R})$ is a jointly continuous version of the local time at point x at time t of $(Y_s)_{s\geq 0}$ (such that, for every t, $\mathcal{L}_t$ is compactly supported).*

Observe that, in every case, there exists $c > 0$ such that

$$a_n \sim c n^2 (\mathbb{E}[R_n])^{\frac{2}{\beta}-2} \tag{7}$$

(see for example [23, p. 36] and [19, pp. 698-703]). It is worth noting that U_n can be rewritten as follows

$$U_n = \sum_{x,y\in\mathbb{Z}^{d_0}} h(\xi_x,\xi_y)N_n(x)N_n(y).$$

Proposition 7. *The results of convergence of finite dimensional distributions of Theorems 3, 5 and 6 hold also if we replace Item (i) of Assumption 1 by the following assumption:*
(i') $\mathbb{E}[\exp(iuh(\xi_1,\xi_1))]-1 = O(|u|^{\beta'})$ for some $\beta' > \beta/2$.

Observe that (i') includes (i) and the case when $h(\xi_1,\xi_1)$ is in the normal domain of attraction of a β'-stable distribution for some $\beta' > \beta/2$, in particular this applies if $h(\xi_1,\xi_1)$ has the same distribution as $h(\xi_1,\xi_2)$.

Proof. Due to Theorems 3, 5 and 6, we know that the finite dimensional distributions of

$$\left(\left(\sum_{x\neq y} h(\xi_x,\xi_y)N_{\lfloor nt\rfloor}(x)N_{\lfloor nt\rfloor}(y)/a_n\right)_{t>0}\right)_n$$

converge. It remains to prove that $(\sum_x h(\xi_x,\xi_x)N^2_{\lfloor nt\rfloor}(x)/a_n)_n$ converges in probability to 0 (for every $t>0$). We write $\varphi_{h(\xi_1,\xi_1)}$ for the characteristic function of $h(\xi_1,\xi_1)$. Let $t>0$ and u be two real numbers. We have

$$\mathbb{E}\left[\exp\left(iu\sum_{x\in\mathbb{Z}^{d_0}}\frac{h(\xi_x,\xi_x)N^2_{\lfloor nt\rfloor}(x)}{a_n}\right)\right] = \mathbb{E}\left[\prod_{x\in\mathbb{Z}^{d_0}}\varphi_{h(\xi_1,\xi_1)}\left(\frac{uN^2_{\lfloor nt\rfloor}(x)}{a_n}\right)\right].$$

To conclude we just have to prove that $\left(\prod_{x\in\mathbb{Z}^{d_0}}\varphi_{h(\xi_1,\xi_1)}\left(\frac{uN^2_{\lfloor nt\rfloor}(x)}{a_n}\right)\right)_n$ converges almost surely to 1. Due to (i'), there exists $C_2>0$ such that we have

$$\left|\prod_{x\in\mathbb{Z}^{d_0}}\varphi_{h(\xi_1,\xi_1)}\left(\frac{uN^2_{\lfloor nt\rfloor}(x)}{a_n}\right)-1\right| \le C_2\sum_{x\in\mathbb{Z}^d}\frac{|u|^{\beta'}N^{2\beta'}_{\lfloor nT\rfloor}(x)}{a_n^{\beta'}}$$

which converges almost surely to 0 since, for every $\varepsilon>0$, the following inequalities hold almost surely, for n large enough

$$R_n \le n^{\frac{1}{\alpha_0}+\varepsilon},\quad N_n^* \le n^{1-\frac{1}{\alpha_0}+\varepsilon} \quad\text{and}\quad a_n^{-1}\le n^{-2+\frac{2}{\alpha_0}-\frac{2}{\alpha_0\beta}+\varepsilon}$$

(see for example [23, 16, 8]). □

3. Examples

The following examples are variants of Example 2.4 from [10]. Observe that

$$\mathbb{P}(h(\xi_1,\xi_2)>z) = \int_E \mathbb{P}(h(x,\xi_2)>z)\,d\mathbb{P}_{\xi_1}(x)$$

and that

$$\mathbb{P}(|h(\xi_1,\xi_2)|>z,|h(\xi_1,\xi_3)|>z') = \int_E \mathbb{P}(|h(x,\xi_2)|>z)\mathbb{P}(|h(x,\xi_2)|>z')\,d\mathbb{P}_{\xi_1}(x).$$

- When $\beta<1$, one can take $E=\mathbb{R}^p$, the distribution of ξ_1 admitting a bounded density f with respect to the Lebesgue measure on E and $h(x,y) = \|x-y\|_\infty^{-p/\beta}\mathbf{1}_{\{x\neq y\}}$. This example fits Assumption 1. Indeed, for every $z>0$, $\mathbb{P}(h(\xi_1,\xi_2)<-z)=0$ and

$$\mathbb{P}(h(x,\xi_2)>z) = \mathbb{P}(\|x-\xi_2\|_\infty < z^{-\frac{\beta}{p}}) \sim_{z\to+\infty} 2^p f(x) z^{-\beta} \quad\text{and}\quad \mathbb{P}(h(x,\xi_2)>z)\le \|f\|_\infty 2^p z^{-\beta}.$$

So

$$\mathbb{P}(h(\xi_1,\xi_2) > z) \sim_{z\to+\infty} 2^p z^{-\beta} \int_{\mathbb{R}^d} (f(x))^2\, dx$$

and

$$\mathbb{P}(|h(\xi_1,\xi_2)| > z, |h(\xi_1,\xi_3)| > z') \le (1 + \|f\|_\infty 2^p)^2 (\max(1,z)\max(1,z'))^{-\beta}.$$

- Analogously, when $\beta \ge 1$, we can take $E = \{\pm 1\} \times \mathbb{R}^p$, $h((\varepsilon,x),(\varepsilon',y)) = \varepsilon\varepsilon' \|x-y\|_\infty^{-p/\beta} \mathbf{1}_{\{x\neq y\}}$ and $\xi_1 = (\varepsilon_1, \vec{\xi_1})$ with ε_1 and $\vec{\xi_1}$ independent; ε_1 being centered and the distribution of $\vec{\xi_1}$ admitting a bounded density f with respect to the Lebesgue measure on $\mathbb{R}^p$. Using the same argument as for the previous example together with Remark 2 we can verify that this example satisfies Assumption 1.

Note that the case $\beta = 1$ contains the more concrete kernel $h(x,y) = 1/(x+y)$ for $x \neq y$ in association with some random variable ξ_1 having a bounded symmetric density on $\mathbb{R}$.

4. Convergence of finite distributions

To simplify notations and the presentation of the proofs, we set

$$|z|_+^\beta := |z|^\beta \quad \text{and} \quad |z|_-^\beta := |z|^\beta \operatorname{sgn}(z) \tag{8}$$

for any real number z. Let $m \ge 1$ and $\theta_1, ..., \theta_m \in \mathbb{R}$ and $0 = t_0 < t_1 < ... < t_m$.

If $\alpha_0 > 1$, we will prove the convergence in distribution of the sequence of random variables

$$\left(a_n^{-1} \sum_{x,y\in\mathbb{Z}^{d_0}} \Big(\sum_{i=1}^m \theta_i N_{\lfloor nt_i \rfloor}(x) N_{\lfloor nt_i \rfloor}(y)) h(\xi_x,\xi_y) \Big) \right)_{n\in\mathbb{N}}. \tag{9}$$

If $\alpha_0 = 1$, since the limit process will have independent increments, it will be more natural to prove the convergence in distribution of the sequence

$$\left(a_n^{-1} \sum_{x,y\in\mathbb{Z}^{d_0}} \left(\sum_{i=1}^m \theta_i \Big(N_{\lfloor nt_i \rfloor}(x) N_{\lfloor nt_i \rfloor}(y) - N_{\lfloor nt_{i-1} \rfloor}(x) N_{\lfloor nt_{i-1} \rfloor}(y) \Big) \right) h(\xi_x,\xi_y) \right)_{n\in\mathbb{N}}.$$

Setting $d_{i,n}(x) := N_{\lfloor nt_i \rfloor}(x) - N_{\lfloor nt_{i-1} \rfloor}(x)$, we observe that

$$\sum_{i=1}^m \theta_i \Big(N_{\lfloor nt_i \rfloor}(x) N_{\lfloor nt_i \rfloor}(y) - N_{\lfloor nt_{i-1} \rfloor}(x) N_{\lfloor nt_{i-1} \rfloor}(y) \Big) = \sum_{i,j=1}^m \theta_{\max(i,j)} d_{i,n}(x) d_{j,n}(y) \tag{10}$$

and hence, if $\alpha_0 = 1$, it is sufficient to study for fixed $\theta_{i,j}$ the sequence of random variables

$$\left(a_n^{-1} \sum_{x,y\in\mathbb{Z}^d} \sum_{i,j=1}^m \theta_{i,j} d_{i,n}(x) d_{j,n}(y) h(\xi_x,\xi_y) \right)_{n\in\mathbb{N}} \tag{11}$$

(in view of applying the results to the particular case when $\theta_{i,j} = \theta_{\max(i,j)}$).

Therefore we have to prove the convergence in distribution of $(a_n^{-1} \sum_{x,y\in\mathbb{Z}^{d_0}} \chi_{n,x,y} h(\xi_x,\xi_y))_n$, with

$$\chi_{n,x,y} := \sum_{i=1}^m \theta_i N_{\lfloor nt_i \rfloor}(x) N_{\lfloor nt_i \rfloor}(y) \quad \text{if} \quad \alpha_0 > 1$$

and

$$\chi_{n,x,y} := \sum_{i,j=1}^m \theta_{i,j} d_{i,n}(x) d_{j,n}(y) \quad \text{if} \quad \alpha_0 = 1.$$

The basic idea is to identify the sequences in (9) and (11) as functionals of some sequence of suitably defined point processes and then to use Kallenberg theorem to prove convergence in law of those point processes. More precisely we will define in section 4.2 the sequence of point processes on $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ defined through

$$\mathcal{N}_n(\tilde{\omega}, \xi) := \sum_{x,y \in \mathbb{Z}^{d_0}} \delta_{a_n^{-1} \zeta_{n,x,y}(\tilde{\omega}) h(\xi_x, \xi_y)},$$

where $(\zeta_{n,x,y})_{n,x,y}$ are suitable random variables defined on some suitable probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that, for every integer n, the random variable $\sum_{x,y \in \mathbb{Z}^{d_0}} \zeta_{n,x,y} h(\xi_x, \xi_y)$ (with respect to $\mathbb{P}_\xi \otimes \tilde{\mathbb{P}}$) has the same law as $\sum_{x,y \in \mathbb{Z}^{d_0}} \chi_{n,x,y} h(\xi_x, \xi_y)$ (with respect to the original probability measure $\mathbb{P}$).

In section 4.1 we prove that the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and the family $(\zeta_{n,x,y})_{n,x,y}$ can be chosen in such a way to satisfy

$$\lim_{n \to +\infty} a_n^{-\beta} \sum_{x,y \in \mathbb{Z}^{d_0}} |\chi_{n,x,y}|_\pm^\beta = \tilde{G}^\pm \quad \text{a.s.}, \tag{12}$$

where $\tilde{G}$ is a suitable random variable on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. The construction will vary depending on whether $\alpha_0 = 1$ or $\alpha_0 > 1$.

The almost sure convergence in (12) will enable us to use Kallenberg theorem in section 4.2 to prove that for almost every $\tilde{\omega} \in \tilde{\Omega}$ the sequence of point processes $(\mathcal{N}_n(\tilde{\omega}, .))_{n \in \mathbb{N}}$ converges in law (with respect to $\mathbb{P}_\xi$) toward a Poisson point process $\mathcal{N}_{\tilde{\omega}}$ on $\mathbb{R}^*$ with the following intensity function

$$z \mapsto \beta |z|^{-\beta-1} \frac{(c_0 + c_1)\tilde{G}^+(\tilde{\omega}) + \operatorname{sgn}(z)(c_0 - c_1)\tilde{G}^-(\tilde{\omega})}{2}.$$

In section 4.3 we will see that $a_n^{-1} \sum_{x,y \in \mathbb{Z}^{d_0}} \zeta_{n,x,y}(\tilde{\omega}) h(\xi_x, \xi_y)$ equals $\int_{\mathbb{R}^*} w \, \mathcal{N}_n(\tilde{\omega}, \xi, dw)$ which as n goes to infinity converges in distribution toward $\int_{\mathbb{R}^*} w \, \mathcal{N}_{\tilde{\omega}}(dw)$. We will also see in section 4.3 that this limit follows a stable law with characteristic function $\Phi_{(c_0+c_1)\tilde{G}^+(\tilde{\omega}), (c_0-c_1)\tilde{G}^-(\tilde{\omega}), \beta}$. This will imply the convergence in distribution of the sequences in (9) and (11) toward the same stable limit.

4.1. **A result of convergence.**

4.1.1. *Case $\alpha_0 = 1$.* We define

$$G_n^\pm := a_n^{-\beta} \sum_{x,y \in \mathbb{Z}^{d_0}} \left| \sum_{i,j=1}^m \theta_{i,j} d_{i,n}(x) d_{j,n}(y) \right|_\pm^\beta \quad \text{and} \quad G^\pm := K_\beta^2 \sum_{i,j=1}^m |\theta_{i,j}|_\pm^\beta (t_i - t_{i-1})(t_j - t_{j-1}), \tag{13}$$

where K_β is the constant defined in Theorems 3 or 5 (depending on whether the random walk $(S_n)_n$ is transient or recurrent with $\alpha = d_0$).

Lemma 8. *If $\alpha_0 = 1$, $(G_n^\pm)_n$ converges almost surely to $G^\pm$.*

Applying this lemma with $\theta_{i,j} = \theta_{\max(i,j)}$, we directly obtain the following almost sure equality

$$\lim_{n \to \infty} a_n^{-\beta} \sum_{x,y \in \mathbb{Z}^{d_0}} \left| \sum_{i=1}^m \theta_i \Big(N_{\lfloor nt_i \rfloor}(x) N_{\lfloor nt_i \rfloor}(y) - N_{\lfloor nt_{i-1} \rfloor}(x) N_{\lfloor nt_{i-1} \rfloor}(y) \Big) \right|_\pm^\beta = K_\beta^2 \sum_{j=1}^m |\theta_j|_\pm^\beta (t_j^2 - t_{j-1}^2). \tag{14}$$

Proof of Lemma 8. We proceed as in [9, 7].

- Let k be a nonnegative integer. Let us prove that

$$\lim_{n \to +\infty} (b_{n,k})^{-2} \sum_{x,y \in \mathbb{Z}^{d_0}} \left(\sum_{i,j=1}^m \theta_{i,j} d_{i,n}(x) d_{j,n}(y) \right)^k = (K_k)^2 \sum_{i,j=1}^m (\theta_{i,j})^k (t_i - t_{i-1})(t_j - t_{j-1}) \ a.s., \tag{15}$$

with $b_{n,k} := n(\log n)^{k-1}$ if $(S_n)_n$ is recurrent (and $\alpha = d_0$) and with $b_{n,k} := n$ if $(S_n)_n$ is transient (extending the definition of K_β given in Theorems 3 or 5 to any nonnegative real number β). Due to [17, p. 10] (transient case) and to [9] (null recurrent case), we know that

$$\forall i \in \{1, ..., m\}, \quad \lim_{n\to\infty} (b_{n,k})^{-1} \sum_{x\in\mathbb{Z}^{d_0}} (d_{i,n}(x))^k = K_k(t_i - t_{i-1}) \quad a.s.. \tag{16}$$

Following some argument from [7], we observe that

$$\left| \sum_{x,y\in\mathbb{Z}^{d_0}} \left(\sum_{i,j=1}^{m} \theta_{i,j} d_{i,n}(x) d_{j,n}(y) \right)^k - \sum_{x,y\in\mathbb{Z}^{d_0}} \sum_{i,j=1}^{m} (\theta_{i,j})^k (d_{i,n}(x) d_{j,n}(y))^k \right|$$

$$\begin{aligned}
&\leq \max_{i,j} |\theta_{i,j}|^k \sum_{((i_1,j_1),...,(i_k,j_k))\in\mathcal{I}} \sum_{x,y\in\mathbb{Z}^{d_0}} \prod_{\ell=1}^{k} d_{i_\ell,n}(x) d_{j_\ell,n}(y) \\
&\leq \max_{i,j} |\theta_{i,j}|^k \left(\sum_{x,y\in\mathbb{Z}^{d_0}} \left(\sum_{i,j=1}^{m} d_{i,n}(x) d_{j,n}(y) \right)^k - \sum_{x,y\in\mathbb{Z}^{d_0}} \sum_{i,j=1}^{m} (d_{i,n}(x) d_{j,n}(y))^k \right) \\
&\leq \max_{i,j} |\theta_{i,j}|^k \left(\left(\sum_{x\in\mathbb{Z}^{d_0}} (N_{\lfloor nt_m \rfloor}(x))^k \right)^2 - \left(\sum_{i=1}^{m} \sum_{x\in\mathbb{Z}^{d_0}} (d_{i,n}(x))^k \right)^2 \right),
\end{aligned}$$

where $\mathcal{I}$ denotes the set of $((i_1, j_1), ..., (i_k, j_k)) \in (\{1, ..., m\}^2)^k$ such that $\#\{(i_1, j_1), ..., (i_k, j_k)\} \geq 2$. Due to (16), we conclude that this term is in $o((b_{n,k})^2)$.

- Assume here that $(S_n)_n$ is recurrent and $\alpha = d_0$. Let us define

$$W_n := \frac{(c_3)^2}{\log^2 n} \sum_{i,j=1}^{m} \theta_{i,j} d_{i,n}(V_n) d_{j,n}(V_n'),$$

with (V_n, V_n') such that the conditional distribution of (V_n, V_n') given S is the uniform distribution on the set $\{z : N_{\lfloor nt_m \rfloor}(z) \geq 1\}^2$. We observe that

$$\mathbb{E}[|W_n|_\pm^u | S] = \frac{c_3^{2u}}{\log^{2u} n} \frac{1}{R^2_{\lfloor nt_m \rfloor}} \sum_{x,y\in\mathbb{Z}^{d_0}} \left| \sum_{i,j=1}^{m} \theta_{i,j} d_{i,n}(x) d_{j,n}(y) \right|_\pm^u \tag{17}$$

for all $u > 0$. Recall that $R_{\lfloor nt_m \rfloor}$ is the cardinal of $\{z : N_{\lfloor nt_m \rfloor}(z) \geq 1\}$ and that $R_n \sim c_3 n / \log n$ a.s.. Due to (15) and since $K_k = \Gamma(k+1)/c_3^{k-1}$, we conclude that, for every non negative integer k, we have, almost surely,

$$\lim_{n\to+\infty} \mathbb{E}[(W_n)^k | S] = (\Gamma(k+1))^2 \sum_{i,j=1}^{m} (\theta_{i,j})^k \frac{t_i - t_{i-1}}{t_m} \frac{t_j - t_{j-1}}{t_m} = \mathbb{E}[W_\infty^k],$$

with $W_\infty = \theta_{V,V'} T T'$ where V', V, T, T' are independent random variables, T and T' having exponential distribution of parameter 1, V and V' being such that $\mathbb{P}(V = i) = \mathbb{P}(V' = i) = \frac{t_i - t_{i-1}}{t_m}$ for every $i \in \{1, \ldots, m\}$. From which we conclude that, almost surely, $(W_n | S)_n$ converges in distribution to W_∞ and that

$$\lim_{n\to+\infty} \mathbb{E}[|W_n|_\pm^\beta | S] = \mathbb{E}[|W_\infty|_\pm^\beta] \quad a.s.. \tag{18}$$

The proof now follows due to (17) and (18).

- Assume now that $(S_n)_n$ is transient and set this time

$$W_n := \sum_{i,j=1}^m \theta_{i,j} d_{i,n}(V_n) d_{j,n}(V'_n),$$

for the same choice of (V_n, V'_n) as in the previous case. Observe that

$$\mathbb{E}[|W_n|_\pm^u|S] = \frac{1}{R^2_{\lfloor nt_m \rfloor}} \sum_{x,y \in \mathbb{Z}^{d_0}} \left| \sum_{i,j=1}^m \theta_{i,j} d_{i,n}(x) d_{j,n}(y) \right|_\pm^u$$

for all $u > 0$. We recall now, that $R_n \sim pn$ with $p := \mathbb{P}(S_k \neq 0,\ \forall k \geq 1) = 2/(\mathbb{E}[N_\infty] + 1)$ (see [23, p. 35]). Due to (15) and since $K_k = \mathbb{E}[N_\infty^{k-1}]$, we obtain that, for every nonnegative integer k, we have almost surely

$$\lim_{n \to +\infty} \mathbb{E}[W_n^k|S] = \left(\frac{\mathbb{E}[N_\infty^{k-1}]}{p} \right)^2 \sum_{i,j=1}^m (\theta_{i,j})^k \frac{t_i - t_{i-1}}{t_m} \frac{t_j - t_{j-1}}{t_m}.$$

So $(W_n|S)_n$ converges in distribution to $TT'\theta_{V,V'}$ where V, V', T, T' are independent random variables such that

$$\forall i \in \{1, ..., m\}, \quad \mathbb{P}(V = i) = \mathbb{P}(V' = i) = \frac{t_i - t_{i-1}}{t_m}$$

and

$$\forall m \geq 1, \quad \mathbb{P}(T = m) = \mathbb{P}(T' = m) = \frac{\mathbb{P}(N_\infty = m)}{mp} = (1-p)^{m-1} p.$$

Indeed, setting $N_\infty(0) := \sup_n N_n(0)$, we have $\mathbb{P}(N_\infty(0) = k) = (1-p)^k p$ for every integer $k \geq 0$. Note that $N_\infty = 1 + N_\infty(0) + \tilde{N}_\infty(0)$ where $\tilde{N}_\infty(0) = \sum_{n \leq -1} \mathbf{1}_{\{S_n = 0\}}$ which is an independent copy of $N_\infty(0)$. Hence we have

$$\mathbb{P}(N_\infty = m) = \sum_{k,\ell \geq 0 : k+\ell = m-1} \mathbb{P}(N_\infty(0) = k) \mathbb{P}(N_\infty(0) = \ell) = mp^2(1-p)^{m-1},$$

for every integer $m \geq 1$. Therefore

$$\lim_{n \to +\infty} \mathbb{E}[|W_n|_\pm^\beta|S] = \left(\frac{\mathbb{E}[N_\infty^{\beta-1}]}{p} \right)^2 \sum_{i,j=1}^m |\theta_{i,j}|_\pm^\beta \frac{t_i - t_{i-1}}{t_m} \frac{t_j - t_{j-1}}{t_m} \quad a.s..$$

This finishes the proof in this case.

□

Since in the main proof we want to treat simultaneously the cases $\alpha_0 = 1$ and $\alpha_0 > 1$, we have to introduce some additional notations which will have its counterpart in the case $\alpha_0 > 1$. So for $\alpha_0 = 1$, we set $\tilde{N}_{n,t_i}(x) := N_{\lfloor nt_i \rfloor}(x)$, $\tilde{N}_n^* := N^*_{\lfloor nt_m \rfloor}$, $\tilde{R}_n := R_{\lfloor nt_m \rfloor}$, $\tilde{G}_n^\pm := G_n^\pm$ and $\tilde{G}^\pm := G^\pm$. We fix $\varepsilon > 0$ such that $\varepsilon < 1/(3 + 4\beta)$ and $(3 + 4\gamma)\varepsilon < \frac{4\gamma}{\beta} - 3$. If $\beta < 4/3$, we assume moreover that $3 - \frac{4\min(1,\gamma)}{\beta} + 7\varepsilon < 0$ (with γ of Item (iv) of Assumption 1). If $\beta \geq 4/3$, we assume that $3 - \frac{4(\theta'+1)}{\beta} + (4\theta' + 7)\varepsilon < 0$ (with θ' of Item (vi) of Assumption 1). We write $\tilde{\mathcal{F}}$ for the sub-algebra generated by S. We consider the set $\tilde{\Omega}_0 \in \tilde{\mathcal{F}}$ on which $(G_n^+, G_n^-, n^{-\varepsilon} N_n^*)$ converges to $(G^+, G^-, 0)$. When $\alpha_0 = 1$, we will make no distinction between $\mathbb{E}$ and $\mathbf{E}$ nor between $\mathbb{P}$ and $\mathbf{P}$.

4.1.2. *Case* $\alpha_0 > 1$. For every $b, t \geq 0$, we set

$$F_{n,t}(b) := n^{-1} \int_0^{n^{\frac{1}{\alpha}} b} N_{\lfloor nt \rfloor}(\lfloor y \rfloor)\, dy, \quad F_{n,t}(-b) := -n^{-1} \int_{-n^{\frac{1}{\alpha}} b}^0 N_{\lfloor nt \rfloor}(\lfloor y \rfloor)\, dy,$$

$$F_t(b) = \int_0^b \mathcal{L}_t(x)\, dx \quad \text{and} \quad F_t(-b) = -\int_{-b}^0 \mathcal{L}_t(x)\, dx,$$

(recall that $\mathcal{L}_s(x)$ is the local time of $(Y_t)_t$ at position x and up to time s). It was proved in [17] that $F_{n,t}(b)$ converges towards $F_t(b)$ in distribution. We prove some vector version of this result. Let us define

$$G_n^{\pm} := a_n^{-\beta} \sum_{x,y \in \mathbb{Z}} \left| \sum_{i=1}^m \theta_i N_{\lfloor nt_i \rfloor}(x) N_{\lfloor nt_i \rfloor}(y) \right|_{\pm}^{\beta} \quad \text{and} \quad G^{\pm} := \int_{\mathbb{R}^2} \left| \sum_{i=1}^m \theta_i \mathcal{L}_{t_i}(x) \mathcal{L}_{t_i}(y) \right|_{\pm}^{\beta} dxdy.$$

Lemma 9. *The finite distributions of* $(F_{n,t_1}, \cdots, F_{n,t_m}, G_n^+, G_n^-)_n$ *converge to the finite distributions of* $(F_{t_1}, \cdots, F_{t_m}, G^+, G^-)$, *i.e.* $((F_{n,t_i}(b_j))_{i=1,\cdots,m, j=1,\dots,q}, G_n^+, G_n^-)_n$ *converges in distribution to the random variable* $((F_{t_i}(b_j))_{i=1,\cdots,m,j=1,\dots,q}, G^+, G^-)_n$, *for every integer* $q \geq 1$ *and every real numbers* $b_1, \dots, b_q$.

Proof. The proof of this convergence result follows mainly the proof of Lemma 6 of [17]. For any real number $\tau > 0$ and any positive integers n and M, we define

$$V^{\pm}(\tau, M, n) := \tau^{2-2\beta} \sum_{|k|,|\ell| \leq M} |T(k, \ell, n)|_{\pm}^{\beta},$$

where

$$T(k, \ell, n) := n^{-2} \sum_{j=1}^m \theta_j \sum_{x=\lceil k\tau n^{\frac{1}{\alpha}} \rceil}^{\lceil (k+1)\tau n^{\frac{1}{\alpha}} \rceil - 1} \sum_{y=\lceil \ell\tau n^{\frac{1}{\alpha}} \rceil}^{\lceil (\ell+1)\tau n^{\frac{1}{\alpha}} \rceil - 1} N_{\lfloor nt_j \rfloor}(x) N_{\lfloor nt_j \rfloor}(y).$$

As in [17], we decompose $G_n^{\pm} - V^{\pm}(\tau, M, n)$ as follows

$$G_n^{\pm} - V^{\pm}(\tau, M, n) = U^{\pm}(\tau, M, n) + W_1^{\pm}(\tau, M, n) + W_2^{\pm}(\tau, M, n),$$

with

$$U^{\pm}(\tau, M, n) := n^{-2\delta\beta} \sum_{(x,y) \in A_{\tau,M,n}} \left| \sum_{j=1}^m \theta_j N_{\lfloor nt_j \rfloor}(x) N_{\lfloor nt_j \rfloor}(y) \right|_{\pm}^{\beta},$$

where $A_{\tau,M,n} := \mathbb{Z}^2 \setminus \{\lceil -M\tau n^{\frac{1}{\alpha}} \rceil, \dots, \lceil (M+1)\tau n^{\frac{1}{\alpha}} \rceil - 1\}^2$,

$$W_1^{\pm}(\tau, M, n) := \sum_{|k|,|\ell| \leq M} \sum_{(x,y) \in E_{k,n} \times E_{\ell,n}} n^{-2\delta\beta} W_{1,k,\ell}^{\pm}(x, y),$$

where $E_{k,n} := \{\lceil k\tau n^{\frac{1}{\alpha}} \rceil, \dots, \lceil (k+1)\tau n^{\frac{1}{\alpha}} \rceil - 1\}$,

$$W_{1,k,\ell}^{\pm}(x, y) := \left| \sum_{j=1}^m \theta_j N_{\lfloor nt_j \rfloor}(x) N_{\lfloor nt_j \rfloor}(y) \right|_{\pm}^{\beta} - n^{2\beta} (\# E_{k,n}\, \# E_{\ell,n})^{-\beta} |T(k, \ell, n)|_{\pm}^{\beta}$$

and

$$W_2^{\pm}(\tau, M, n) := \sum_{|k|,|\ell| \leq M} \left\{ n^{2\beta - 2\delta\beta} (\# E_{k,n}\, \# E_{\ell,n})^{1-\beta} - \tau^{2-2\beta} \right\} |T(k, \ell, n)|_{\pm}^{\beta}.$$

The proof follows now in five steps:

1) Observe that, due to [17, Lemma 1], there exists a function η satisfying $\lim_{x \to +\infty} \eta(x) = 0$ such that

$$\sup_n \mathbb{P}\Big(U^{\pm}(\tau, M, n) \neq 0 \Big) \leq \sup_n \mathbb{P}\Big(\exists x \ :\ |x| \geq M\tau n^{\frac{1}{\alpha}} \text{ and } N_{\lfloor nt_m \rfloor}(x) \neq 0 \Big) = \eta(M\tau). \tag{19}$$

2) We prove that there exists some $K>0$ and $u>0$ such that for all $M>1$ one has

$$\sup_n \mathbb{E}[|W_1^{\pm}(\tau,M,n)|] \le K(M\tau)^2\tau^u. \tag{20}$$

We first do the case $\beta\le 1$. Using the fact that $||a|_\pm^\beta - |b|_\pm^\beta| \le 2^{1-\beta}|a-b|^\beta$, we have

$$\begin{aligned}
&2^{\beta-1}\mathbb{E}[|W_{1,k,\ell}^{\pm}(x,y)|]\\
\le\ & \mathbb{E}\left[\left|\sum_{j=1}^m \theta_j N_{\lfloor nt_j\rfloor}(x)N_{\lfloor nt_j\rfloor}(y) - n^2(\#E_{k,n}\,\#E_{\ell,n})^{-1}T(k,\ell,n)\right|^\beta\right]\\
\le\ & \left\|\sum_{j=1}^m \theta_j N_{\lfloor nt_j\rfloor}(x)N_{\lfloor nt_j\rfloor}(y) - n^2(\#E_{k,n}\,\#E_{\ell,n})^{-1}T(k,\ell,n)\right\|_2^\beta\\
\le\ & (\#E_{k,n}\,\#E_{\ell,n})^{-\beta}\left\|\sum_{j=1}^m \sum_{(x',y')\in E_{k,n}\times E_{\ell,n}} \theta_j\left(N_{\lfloor nt_j\rfloor}(x)N_{\lfloor nt_j\rfloor}(y) - N_{\lfloor nt_j\rfloor}(x')N_{\lfloor nt_j\rfloor}(y')\right)\right\|_2^\beta\\
\le\ & (\#E_{k,n}\,\#E_{\ell,n})^{-\frac{\beta}{2}}\left(\sum_{i=1}^m\theta_i^2\sum_{j=1}^m \sum_{(x',y')\in E_{k,n}\times E_{\ell,n}} \left\|\left(N_{\lfloor nt_j\rfloor}(x)N_{\lfloor nt_j\rfloor}(y) - N_{\lfloor nt_j\rfloor}(x')N_{\lfloor nt_j\rfloor}(y')\right)\right\|_2^2\right)^{\frac{\beta}{2}},
\end{aligned}$$

due to the Cauchy-Schwarz inequality. Now we have to estimate

$$\sum_{(x',y')\in E_{k,n}\times E_{\ell,n}} \mathbb{E}\Big[|N_{\lfloor nt_j\rfloor}(x)N_{\lfloor nt_j\rfloor}(y) - N_{\lfloor nt_j\rfloor}(x')N_{\lfloor nt_j\rfloor}(y')|^2\Big],$$

for $(x,y)\in E_{k,n}\times E_{\ell,n}$. To this end, we use $\mathbb{E}[|ab-a'b'|^2] \le 2\|a\|_4^2\|b-b'\|_4^2 + \|a-a'\|_4^2\|b'\|_4^2$ together with the fact that

$$\sup_x \mathbb{E}[(N_n(x))^4] = O(n^{4-\frac{4}{\alpha}}) \quad\text{and}\quad \sup_{y\ne z}\frac{\mathbb{E}[|N_n(y)-N_n(z)|^4]}{|y-z|^{2\alpha-2}} = O(n^{2-\frac{2}{\alpha}})$$

(see for example [16, p.77] for the last estimate). This gives,

$$\mathbb{E}[|N_{\lfloor nt_j\rfloor}(x)N_{\lfloor nt_j\rfloor}(y) - N_{\lfloor nt_j\rfloor}(x')N_{\lfloor nt_j\rfloor}(y')|^2] \le C\tau^{\alpha-1}n^{4-\frac{4}{\alpha}}, \tag{21}$$

for every $(x,y),(x',y')\in E_{k,n}\times E_{\ell,n}$ and for some $C>0$ independent of (τ,M,n,k,ℓ). Therefore, we obtain

$$\mathbb{E}[|W_1^{\pm}(\tau,M,n)|] \le C'(2M+1)^2\tau^{2+\frac{\beta}{2}(\alpha-1)},$$

where C' does not depend on (τ,M,n). From this we conclude in the case $\beta\le 1$.

When $\beta>1$, we use $||a|_\pm^\beta - |b|_\pm^\beta| \le \beta|a-b|(|a|^{\beta-1}+|b|^{\beta-1})$ combined with the Cauchy-Schwarz inequality and obtain

$$\begin{aligned}
&\mathbb{E}[|W^{\pm}_{1,k,\ell}(x,y)|]\\
\leq\ &\beta(\#E_{k,n}\,\#E_{\ell,n})^{-1}\left\|\sum_{j=1}^m\theta_j\sum_{(x',y')\in E_{k,n}\times E_{\ell,n}}(N_{\lfloor nt_j\rfloor}(x)N_{\lfloor nt_j\rfloor}(y)-N_{\lfloor nt_j\rfloor}(x')N_{\lfloor nt_j\rfloor}(y'))\right\|_2\times\\
&\times\left\|\left|\sum_{j=1}^m\theta_jN_{\lfloor nt_j\rfloor}(x)N_{\lfloor nt_j\rfloor}(y)\right|^{\beta-1}+\Big(n^2(\#E_{k,n}\,\#E_{\ell,n})^{-1}|T(k,\ell,n)|\Big)^{\beta-1}\right\|_2\\
\leq\ &\beta(\#E_{k,n}\,\#E_{\ell,n})^{-1}\sum_{j=1}^m|\theta_j|\sum_{(x',y')\in E_{k,n}\times E_{\ell,n}}\left\|(N_{\lfloor nt_j\rfloor}(x)N_{\lfloor nt_j\rfloor}(y)-N_{\lfloor nt_j\rfloor}(x')N_{\lfloor nt_j\rfloor}(y'))\right\|_2\times\\
&\times\left(\left\|\sum_{j=1}^m\theta_jN_{\lfloor nt_j\rfloor}(x)N_{\lfloor nt_j\rfloor}(y)\right\|_{2(\beta-1)}^{\beta-1}+n^{2\beta-2}(\#E_{k,n}\,\#E_{\ell,n})^{1-\beta}\,\|T(k,\ell,n)\|_{2(\beta-1)}^{\beta-1}\right)\\
\leq\ &C(\tau^{\alpha-1}n^{4-\frac{4}{\alpha}})^{\frac12}\left(\sup_{x'}\|N_{\lfloor nt_m\rfloor}(x')\|^{2\beta-2}_{4(\beta-1)}+\right.\\
&\left.+n^{2\beta-2}(\tau n^{\frac1\alpha})^{2-2\beta}\left(n^{-2}(\tau n^{\frac1\alpha})^2\sup_{x'}\|N_{\lfloor nt_m\rfloor}(x')\|^2_{4(\beta-1)}\right)^{\beta-1}\right),
\end{aligned}$$

due to the Cauchy Schwarz inequality and to (21). Hence we have

$$\mathbb{E}[|W^{\pm}_{1,k,\ell}(x,y)|]\leq C'\tau^{\frac{\alpha-1}{2}}n^{2-\frac{2}{\alpha}}n^{\left(1-\frac1\alpha\right)2(\beta-1)}=C'\tau^{\frac{\alpha-1}{2}}n^{2\beta\left(1-\frac1\alpha\right)}$$

for some $C'>0$ and so

$$\mathbb{E}[|W^{\pm}_1(\tau,M,n)|]\leq C''(2M+1)^2\tau^{2+\frac{\alpha-1}{2}},$$

where C'' does not depend on (τ,M,n) and we conclude in the case when $\beta>1$.

3) We notice that

$$\mathcal{V}^{\pm}(\tau,M):=\tau^{2-2\beta}\sum_{|k|,|\ell|\leq M}\left|\int_{k\tau}^{(k+1)\tau}\int_{\ell\tau}^{(\ell+1)\tau}\sum_{j=1}^m\theta_j\mathcal{L}_{t_j}(x)\mathcal{L}_{t_j}(y)\,dxdy\right|_{\pm}^{\beta}$$

converge to $G^{\pm}$ as $(\tau,M\tau)\to(0,\infty)$, since the local times $x\mapsto\mathcal{L}_{t_j}(x)$ are almost surely continuous and compactly supported (see [17]).

4) We observe that, for every choice of (τ,M) the sequence $(W_2^{\pm}(\tau,M,n))_n$ converges in probability to 0 as $n\to\infty$. This comes from the fact that for every (k,ℓ) the sequence $(T(k,\ell,n))_n$ converges in distribution to

$$\sum_{j=1}^m\theta_j\int_{k\tau}^{(k+1)\tau}\mathcal{L}_{t_j}(x)\,dx\int_{\ell\tau}^{(\ell+1)\tau}\mathcal{L}_{t_j}(y)\,dy$$

and the fact that the sequence $(n^{2\beta(1-\delta)}(\#E_{k,n}\,\#E_{\ell,n})^{1-\beta}-\tau^{2-2\beta})_n$ converges to 0.

5) For every choice of (τ,M), for every q and every real numbers $b_1,...,b_q$, the sequence of random variables $((F_{n,t_i}(b_j))_{i,j},V^+(\tau,M,n),V^-(\tau,M,n)))_n$ converges in distribution to $((F_{t_i}(b_j))_{i,j},\mathcal{V}^+(\tau,M),\mathcal{V}^-(\tau,M))$. Indeed, we recall that

$$V^{\pm}(\tau,M,n):=\tau^{2-2\beta}\sum_{|k|,|\ell|\leq M}|T(k,\ell,n)|^{\beta}_{\pm}$$

and notice that

$$T(k,\ell,n)=\left(\sum_{j=1}^m\theta_j\Big(F_{n,t_j}((k+1)\tau)-F_{n,t_j}(k\tau)\Big)\Big(F_{n,t_j}((\ell+1)\tau)-F_{n,t_j}(\ell\tau)\Big)\right)+O(n^{-1})$$

6) Now we conclude. Let $z_{i,j},z_\pm\in\mathbb{R}$ and $\epsilon>0$. Due to Points 1, 2 and 3, we fix $M>1$ and $\tau>0$ such, for every n, we have

$$\mathbb{E}\left[\left|e^{i(z_+G_n^++z_-G_n^-)}-e^{i(z_+(V^+(\tau,M,n)+W_2^+(\tau,M,n))+z_-(V^-(\tau,M,n)+W_2^-(\tau,M,n)))}\right|\right]<\varepsilon \tag{22}$$

and

$$\mathbb{E}\left[\left|e^{i\left(z_+\mathcal{V}^+(\tau,M)+z_-\mathcal{V}^-(\tau,M)\right)}-e^{i\left(z_+G^++z_-G^-\right)}\right|\right]<\epsilon. \tag{23}$$

Due to Points 4 and 5 for this choice of (M,τ), there exists n_0 such that for every $n\ge n_0$,

$$\mathbb{E}\left[\left|e^{iz_+W_2^+(\tau,M,n)+iz_-W_2^-(\tau,M,n)}-1\right|\right]<\epsilon \tag{24}$$

and

$$\left|\mathbb{E}\left[e^{i\left(\sum_{i,j}z_{ij}F_{t_i}(b_j))+z_+\mathcal{V}^+(\tau,M)+z_-\mathcal{V}^-(\tau,M)\right)}\right]-\mathbb{E}\left[e^{i\left(\sum_{i,j}z_{ij}F_{n,t_i}(b_j))+z_+V^+(\tau,M,n)+z_-V^-(\tau,M,n)\right)}\right]\right|<\epsilon. \tag{25}$$

Hence, for every $n\ge n_0$, we have

$$\begin{aligned}&\left|\mathbb{E}\left[e^{i\left(\sum_{i,j}z_{ij}F_{t_i}(b_j))+z_+G^++z_-G^-\right)}\right]-\mathbb{E}\left[e^{i\left(\sum_{i,j}z_{ij}F_{n,t_i}(b_j))+z_+G_n^++z_-G_n^-\right)}\right]\right|\\ \le\ & 3\epsilon+\left|\mathbb{E}\left[e^{i\left(\sum_{i,j}z_{ij}F_{t_i}(b_j))+z_+\mathcal{V}^+(\tau,M)+z_-\mathcal{V}^-(\tau,M)\right)}\right]-\mathbb{E}\left[e^{i\left(\sum_{i,j}z_{ij}F_{n,t_i}(b_j))+z_+V^+(\tau,M,n)+z_-V^-(\tau,M,n)\right)}\right]\right|\\ \le\ & 4\varepsilon.\end{aligned}$$

where we used (22), (23), (24) for the first inequality and (25) for the last one. ☐

Let $\mathcal{C}$ be the set of continuous functions $g:\mathbb{R}\to[-t_m,t_m]$. We endow this set with the following metric D corresponding to the uniform convergence on every compact:

$$D(g,h):=\sum_{N\ge1}2^{-N}\sup_{x\in[-N;N]}|g(x)-h(x)|.$$

Lemma 10. *The sequence $(F_{n,t_1},...,F_{n,t_m})_{n\in\mathbb{N}}$ is tight in $(\mathcal{C},D)^m$.*

Proof. It is enough to prove the tightness of F_{n,t_i} for all $i\in\{1,...,m\}$. To simplify notations in this proof we use F_n to denote $F_{n,t_i}/t_i$ and F to denote F_{t_i}/t_i. As usual, for any $f\in\mathcal{C}$, we denote by $\omega(f,\cdot)$ the modulus of continuity of f. Since $F_n(0)=0$ for every n, it is enough to prove

$$\forall\varepsilon>0,\quad \lim_{\delta\to0}\limsup_{n\to+\infty}\mathbb{P}(\omega(F_n,\delta)\ge\varepsilon)=0 \tag{26}$$

(see [2, p.83]). Let $\varepsilon>0$ and $\varepsilon_0>0$. Let $M>0$ be such that $\mathbb{P}\Big(|F(M)-F(-M)|\le1-(\varepsilon/2)\Big)\le\varepsilon_0/2$. Since $(F_n(M)-F_n(-M))_n$ converges in distribution to $F(M)-F(-M)$, we have

$$\limsup_{n\to+\infty}\mathbb{P}\Big(|F_n(M)-F_n(-M)|\le1-(\varepsilon/2)\Big)\le\mathbb{P}\Big(|F(M)-F(-M)|\le1-(\varepsilon/2)\Big)\le\varepsilon_0/2. \tag{27}$$

Let $\delta_0 > 0$ be such that, for every $\delta \in (0, \delta_0)$, $\mathbb{P}(\omega(F, \delta) \geq \varepsilon/2) \leq \varepsilon_0/2$ (since F is almost surely uniformly continuous). Since the finite distributions of $(F_n)_n$ converge to the finite distribution of F, we have

$$\begin{aligned}
(28) \qquad & \limsup_{n\to+\infty} \mathbb{P}\left(\exists k = -\left\lceil\frac{M}{\delta}\right\rceil, ..., \left\lceil\frac{M}{\delta}\right\rceil,\ |F_n(k\delta) - F_n((k+1)\delta)| \geq \frac{\varepsilon}{2}\right) \\
\leq\ & \mathbb{P}\left(\exists k = -\left\lceil\frac{M}{\delta}\right\rceil, ..., \left\lceil\frac{M}{\delta}\right\rceil,\ |F(k\delta) - F((k+1)\delta)| \geq \frac{\varepsilon}{2}\right) \\
\leq\ & \mathbb{P}\left(\omega(F, \delta) \geq \frac{\varepsilon}{2}\right) \leq \frac{\varepsilon_0}{2}.
\end{aligned}$$

Putting (27) and (28) together, we obtain that, for every $\delta < \delta_0$, we have

$$\begin{aligned}
\limsup_{n\to+\infty} \mathbb{P}(\omega(F_n, \delta) \geq \varepsilon) \quad \leq \quad & \limsup_{n\to+\infty} \mathbb{P}\left(\exists k = -\left\lceil\frac{M}{\delta}\right\rceil, ..., \left\lceil\frac{M}{\delta}\right\rceil,\ |F_n(k\delta) - F_n((k+1)\delta)| \geq \frac{\varepsilon}{2}\right) \\
& + \limsup_{n\to+\infty} \mathbb{P}\Big(|F_n(M) - F_n(-M)| \leq 1 - (\varepsilon/2)\Big)
\end{aligned}$$

and so

$$\limsup_{n\to+\infty} \mathbb{P}(\omega(F_n, \delta) \geq \varepsilon) \leq \varepsilon_0.$$

□

Due to Lemma 9 and Lemma 10, the sequence $(F_{n,t_1}, \ldots, F_{n,t_m}, G_n^+, G_n^-)_n$ converges in distribution to $(F_{t_1}, \ldots, F_{t_m}, G^+, G^-)$ in $(\mathcal{C}, D)^m \times (\mathbb{R}, |\cdot|)^2$.

We fix $\varepsilon \in (0, \beta\delta/(1+\beta))$ such that $(3+4\beta)\varepsilon < 1/\alpha$ and $(3+4\gamma)\varepsilon\alpha < \frac{4\gamma}{\beta} - 3$ (this is possible due to $\gamma > 3\beta/4$). If $\beta < 4/3$, we assume moreover that $\frac{3}{\alpha} - \frac{4\min(1,\gamma)}{\alpha\beta} + 7\varepsilon < 0$ (with γ of Item (iv) of Assumption 1). If $\beta \geq 4/3$, we assume also that $\frac{1}{\alpha}\left(3 - \frac{4(\theta'+1)}{\beta}\right) + (4\theta' + 7)\varepsilon < 0$ (with θ' of Item (vi) of Assumption 1). Using for example [16] for the maximal occupation time and appendix of [8] for the range, we know that $(n^{-1/\alpha-\varepsilon} R_n, n^{(1/\alpha)-1-\varepsilon} N_n^*)_n$ converges almost surely to 0. Therefore the sequence $(F_{n,t_1}, \ldots, F_{n,t_m}, G_n^+, G_n^-, n^{-1/\alpha-\varepsilon} R_n, n^{(1/\alpha)-1-\varepsilon} N_n^*)_n$ converges in distribution to $(F_{t_1}, \ldots, F_{t_m}, G^+, G^-, 0, 0)$ in $(\mathcal{C}, D)^m \times (\mathbb{R}, |\cdot|)^4$.

Now using the Skorokhod representation theorem (see [12] p.1569) (since $(\mathcal{C}, D)$ and $\mathbb{R}$ are separable and complete), we know that there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with random variables

$$(\tilde{F}_{n,t_1}, \ldots, \tilde{F}_{n,t_m}, \tilde{G}_n^+, \tilde{G}_n^-, \tilde{R}_n, \tilde{N}_n^*)_n \text{ and } (\tilde{F}_{t_1}, \ldots, \tilde{F}_{t_m}, \tilde{G}^+, \tilde{G}^-)$$

defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that

- for every integer n, $(\tilde{F}_{n,t_1}, \ldots, \tilde{F}_{n,t_m}, \tilde{G}_n^+, \tilde{G}_n^-, \tilde{R}_n, \tilde{N}_n^*)$ has the same distribution (with respect to $\tilde{\mathbb{P}}$) as $(F_{n,t_1}, \ldots, F_{n,t_m}, G_n^+, G_n^-, R_{\lfloor nt_m\rfloor}, N^*_{\lfloor nt_m\rfloor})$ (with respect to $\mathbb{P}$) in $(\mathcal{C}, D)^m \times (\mathbb{R}, |\cdot|)^4$;
- $(\tilde{F}_{t_1}, \ldots, \tilde{F}_{t_m}, \tilde{G}^+, \tilde{G}^-)$ has the same distribution as $(F_{t_1}, \ldots, F_{t_m}, G^+, G^-)$ in $(\mathcal{C}, D)^m \times (\mathbb{R}, |\cdot|)^4$;
- the sequence $(\tilde{F}_{n,t_1}, \ldots, \tilde{F}_{n,t_m}, \tilde{G}_n^+, \tilde{G}_n^-, n^{-1/\alpha-\varepsilon}\tilde{R}_n, n^{(1/\alpha)-1-\varepsilon}\tilde{N}_n^*)_n$ converges almost surely to $(\tilde{F}_{t_1}, \ldots, \tilde{F}_{t_m}, \tilde{G}^+, \tilde{G}^-, 0, 0)$ in $(\mathcal{C}, D)^m \times (\mathbb{R}, |\cdot|)^4$.

Observe that, for every $x \in \mathbb{Z}$ and every $n \geq 1$, $\mathfrak{N}_n(x) : f \mapsto n(f((x+1)n^{-\frac{1}{\alpha}}) - f(xn^{-\frac{1}{\alpha}}))$ is a continuous functional of $(\mathcal{C}, D)$ and that $N_{\lfloor nt_i\rfloor}(x) = \mathfrak{N}_n(x)(F_{n,t_i})$ (for every $i \in \{1, \ldots, m\}$). Therefore, for every integers x and $n \geq 1$, for every $i \in \{1, \ldots, m\}$, we define

$$\tilde{N}_{n,t_i}(x) := \mathfrak{N}_n(x)(\tilde{F}_{n,t_i}).$$

Observe that, for every integer $N \geq 1$,

$$\left((\tilde{N}_{n,t_i}(x))_{x\in\{-N,\dots,N\};i\in\{1,\dots,m\}}, \tilde{N}_n^*, \tilde{R}_n, \tilde{G}_n^{\pm}\right)$$

has the same distribution as

$$\left((N_{\lfloor nt_i\rfloor}(x))_{x\in\{-N,\dots,N\};i\in\{1,\dots,m\}}, N^*_{\lfloor nt_m\rfloor}, R_{\lfloor nt_m\rfloor}, G_n^{\pm}\right).$$

In particular $\tilde{N}_{n,t_i}(x)$ takes integer values and $0 \leq \tilde{N}_{n,t_i}(x) \leq \tilde{N}_{n,t_m}(x)$. Moreover we have the following result.

Lemma 11. *Let n be a positive integer. We have*

$$\sup_{x\in\mathbb{Z}} \tilde{N}_{n,t_m}(x) \leq \tilde{N}_n^*, \tag{29}$$

$$\#\{x \in \mathbb{Z} : \tilde{N}_{n,t_m}(x) > 0\} = \tilde{R}_n \tag{30}$$

and

$$\tilde{G}_n^{\pm} = n^{-2\beta\delta} \sum_{x,y\in\mathbb{Z}} \left|\sum_{i=1}^m \theta_i \tilde{N}_{n,t_i}(x)\tilde{N}_{n,t_i}(y)\right|_{\pm}^{\beta}. \tag{31}$$

Proof. (29) comes from the fact that, for every integers x and $n \geq 1$, $\tilde{N}_n^* - \tilde{N}_{n,t_m}(x)$ has the same distribution as $N^*_{\lfloor nt_m\rfloor} - N_{\lfloor nt_m\rfloor}(x)$ which is non negative.

To prove (30), we observe that

$$\tilde{R}_n - \#\{x \in \mathbb{Z} \ : \ \tilde{N}_{n,t_m}(x) > 0\} = \lim_{N\to+\infty} \left(\tilde{R}_n - \#\big\{x \in \{-N,\dots,N\} \ : \ \tilde{N}_{n,t_m}(x) > 0\big\}\right).$$

But, for every $N \geq 1$, $\tilde{R}_n - \#\big\{x \in \{-N,\dots,N\} \ : \ \tilde{N}_{n,t_m}(x) > 0\big\}$ has the same distribution as $R_{\lfloor nt_m\rfloor} - \#\big\{x \in \{-N,\dots,N\} \ : \ N_{\lfloor nt_m\rfloor}(x) > 0\big\}$ which converges to 0 as N goes to infinity. This gives (30) by uniqueness of the limit for the convergence in probability.

Finally, we observe that $\tilde{G}_n^{\pm} - n^{-2\beta\delta}\sum_{x,y\in\mathbb{Z}} \left|\sum_{i=1}^m \theta_i \tilde{N}_{n,t_i}(x)\tilde{N}_{n,t_i}(y)\right|_{\pm}^{\beta}$ is the limit as N goes to infinity of

$$\tilde{G}_n^{\pm} - n^{-2\beta\delta} \sum_{|x|,|y|\leq N} \left|\sum_{i=1}^m \theta_i \tilde{N}_{n,t_i}(x)\tilde{N}_{n,t_i}(y)\right|_{\pm}^{\beta}$$

which has the same distribution as

$$G_n^{\pm} - n^{-2\beta\delta} \sum_{|x|,|y|\leq N} \left|\sum_{i=1}^m \theta_i N_{\lfloor nt_i\rfloor}(x)N_{\lfloor nt_i\rfloor}(y)\right|_{\pm}^{\beta}.$$

But this last random variable converges to 0 as N goes to infinity and we obtain (31). □

Let us write $(\Omega, \mathcal{F}, \mathbb{P})$ for the original space on which ξ and S are defined. We denote $\mathcal{F}_\xi$ for the sub-σ-algebra of $\mathcal{F}$ generated by ξ and $\mathbb{P}_\xi$ for the restriction of $\mathbb{P}$ to $\mathcal{F}_\xi$. Now we define $(\mathbf{\Omega}, \mathcal{T}, \mathbf{P})$ as the direct product of $(\Omega, \mathcal{F}_\xi, \mathbb{P}_\xi)$ with $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. We observe that $\mathbb{P}_\xi(\cdot) = \mathbf{P}(\cdot|\tilde{\mathcal{F}})$.

Lemma 12. *For every integer $n \geq 1$, the random variable $\tilde{\mathfrak{A}}_n := \sum_{x,y\in\mathbb{Z}}\sum_{i=1}^m \theta_i \tilde{N}_{n,t_i}(x)\tilde{N}_{n,t_i}(y)h(\xi_x, \xi_y)$ has the same distribution (with respect to $\mathbf{P}$) as $\mathfrak{A}_n := \sum_{x,y\in\mathbb{Z}}\sum_{i=1}^m \theta_i N_{\lfloor nt_i\rfloor}(x)N_{\lfloor nt_i\rfloor}(y)h(\xi_x, \xi_y)$ (with respect to $\mathbb{P}$).*

Proof. We proceed as in the proof of Lemma 11. Observe that $\tilde{\mathfrak{A}}_n$ is the limit as N goes to infinity of $\tilde{\mathfrak{A}}_{n,N} := \sum_{|x|,|y|\le N}\sum_{i=1}^m \theta_i \tilde{N}_{n,t_i}(x)\tilde{N}_{n,t_i}(y)h(\xi_x,\xi_y)$ which has the same distribution as $\mathfrak{A}_{n,N} := \sum_{|x|,|y|\le N}\sum_{i=1}^m \theta_i N_{\lfloor nt_i\rfloor}(x)N_{\lfloor nt_i\rfloor}(y)h(\xi_x,\xi_y)$. But $\mathfrak{A}_n = \lim_{N\to+\infty}\mathfrak{A}_{n,N}$. We conclude by unicity of the limit for the convergence in distribution. □

Let $\tilde{\Omega}_0 \subset \tilde{\Omega}$ be the set of $\tilde{\mathbb{P}}$-measure one on which $(\tilde{F}_{n,t_1},\dots,\tilde{F}_{n,t_m},\tilde{G}_n^+,\tilde{G}_n^-,n^{-1/\alpha-\varepsilon}\tilde{R}_n,n^{(1/\alpha)-1-\varepsilon}\tilde{N}_n^*)_n$ converges to $(\tilde{F}_{t_1},\dots,\tilde{F}_{t_m},\tilde{G}^+,\tilde{G}^-,0,0)$ in $\mathcal{C}^m\times\mathbb{R}^4$.

4.2. **A conditional limit theorem for some associated point process.** To simplify notations, we set

$$\zeta_{n,x,y} := \sum_{i=1}^m \theta_i \tilde{N}_{n,t_i}(x)\tilde{N}_{n,t_i}(y) \quad \text{if} \quad \alpha_0 > 1 \tag{32}$$

and

$$\zeta_{n,x,y} := \sum_{i,j=1}^m \theta_{i,j} d_{i,n}(x)d_{j,n}(y) \quad \text{if} \quad \alpha_0 = 1. \tag{33}$$

With these notations we have

$$\tilde{G}_n^{\pm} = a_n^{-\beta}\sum_{x,y}|\zeta_{n,x,y}|_{\pm}^{\beta}.$$

For every $\tilde{\omega}\in\tilde{\Omega}_0$, we consider the point process $\mathcal{N}_n$ on $\mathbb{R}^*$ defined by

$$\mathcal{N}_n(\tilde{\omega},\xi)(dz) := \sum_{x,y\in\mathbb{Z}:x\neq y}\delta_{a_n^{-1}\zeta_{n,x,y}(\tilde{\omega})h(\xi_x,\xi_y)}(dz).$$

We already mentioned in (7) that $a_n \sim cn^2(\mathbb{E}[R_n])^{\frac{2}{\beta}-2}$ for some $c>0$ and observe that in any case

$$\forall\gamma_0>0,\quad a_n^{-1} = o\left(n^{-2+\frac{2}{\alpha_0}-\frac{2}{\alpha_0\beta}+\gamma_0}\right). \tag{34}$$

Moreover note that for the $\epsilon>0$ which was fixed in the previous subsection we have

$$n^{\frac{1}{\alpha_0}-1-\epsilon}\tilde{N}_n^* \xrightarrow{a.s.} 0$$

and

$$n^{-\frac{1}{\alpha_0}-\epsilon}\tilde{R}_n \xrightarrow{a.s.} 0.$$

In the following we will prove that the sequence of point processes $\mathcal{N}_n; n\in\mathbb{N}$ converges toward some Poisson point process for $\tilde{\mathbb{P}}$ almost all $\tilde{\omega}\in\tilde{\Omega}$. We will essentially follow the notation from [21] and denote by $M_p(\mathbb{R}^*)$ the set of point measures on $\mathbb{R}^*$. Further, $\mathcal{M}_p(\mathbb{R}^*)$ is the smallest σ-algebra containing all sets A of the form

$$A = \{m\in M_p(\mathbb{R}^*); m(F)\in B\}$$

for some $F\in\mathcal{B}(\mathbb{R}^*)$ and $B\in\mathcal{B}([0,\infty])$. We introduce the following metric on $\mathbb{R}^*$

$$d(x,y) := \begin{cases} |\log(x/y)| & \text{if } \operatorname{sgn}(x)=\operatorname{sgn}(y); \\ |\log|x|| + |\log|y|| + 1 & \text{if } \operatorname{sgn}(x)\neq\operatorname{sgn}(y). \end{cases}$$

With this metric $\mathbb{R}^*$ becomes a complete separable metric space. We will denote by $C_K(\mathbb{R}^*)$ the space of continuous functions $f:\mathbb{R}^*\to\mathbb{R}$ with compact support with respect to this metric. A sequence of Radon measures μ_n is said to converge with respect to the vague topology toward some Radon measure μ if for all $f\in C_K(\mathbb{R}^*)$ one has

$$\lim_{n\to\infty}\int_{\mathbb{R}^*} f\,d\mu_n = \int_{\mathbb{R}^*} f\,d\mu.$$

It is well known that the vague topology on the Radon measures can be generated by some metric which turns it into a complete metric space (see [21] p.147) and that the set of point measures is closed in the vague topology (see [21] p.145). We will say that a sequence of point processes $\mathcal{N}_n; n \in \mathbb{N}$ converges in distribution toward a Point process $\mathcal{N}$ if for all bounded vaguely continuous functions $F : M_p(\mathbb{R}^*) \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \mathbb{E}[F(\mathcal{N}_n)] = \mathbb{E}[F(\mathcal{N})].$$

Proposition 13. *For every $\tilde{\omega} \in \tilde{\Omega}_0$, $\mathcal{N}_n(\tilde{\omega}, \cdot)$ converges in distribution (with respect to $\mathbb{P}_\xi$) to a Poisson process $\mathcal{N}_{\tilde{\omega}}$ on $\mathbb{R} \setminus \{0\}$ of intensity $\eta_{\tilde{\omega}}$ given by*

$$\eta_{\tilde{\omega}}([d, d')) = (d^{-\beta} - d'^{-\beta})\frac{(c_0 + c_1)\tilde{G}^+(\tilde{\omega}) + (c_0 - c_1)\tilde{G}^-(\tilde{\omega})}{2},$$

and

$$\eta_{\tilde{\omega}}((-d', -d]) = (d^{-\beta} - d'^{-\beta})\frac{(c_0 + c_1)\tilde{G}^+(\tilde{\omega}) - (c_0 - c_1)\tilde{G}^-(\tilde{\omega})}{2},$$

(with convention $\infty^{-\beta} = 0$) for every $0 < d < d' \leq +\infty$.

Proof. Our proof is based on some method presented in [10]. Due to Kallenberg's theorem [21], it is enough to prove that, for any finite union $R = \bigcup_{i=1}^K Q_i$ of intervals, where $Q_i := [d_i, d'_i) \subset (0, +\infty)$ or $Q_i = (-d'_i, -d_i] \subset (-\infty, 0)$. We have

$$\lim_{n\to+\infty} \mathbf{E}[\mathcal{N}_n(R)|\tilde{\mathcal{F}}](\tilde{\omega}) = \eta_{\tilde{\omega}}(R) \tag{35}$$

and

$$\lim_{n\to+\infty} \mathbf{P}(\mathcal{N}_n(R) = 0|\tilde{\mathcal{F}})(\tilde{\omega}) = e^{-\eta_{\tilde{\omega}}(R)}. \tag{36}$$

We start with the proof of (35). By linearity, it is enough to prove it for a single interval Q. For any interval $Q = [d, d') \subset (0, +\infty)$, since ξ is a sequence of iid random variables, we have

$$\mathbf{E}[\mathcal{N}_n(Q)|\tilde{\mathcal{F}}] = \sum_{x,y\in\mathbb{Z}^{d_0}:x\neq y} \Big(\mathbf{P}(\mathcal{A}_{n,x,y}|\tilde{\mathcal{F}})\mathbf{1}_{\{\zeta_{n,x,y}>0\}} + \mathbf{P}(\mathcal{B}_{n,x,y}|\tilde{\mathcal{F}})\mathbf{1}_{\{\zeta_{n,x,y}<0\}}\Big),$$

with

$$\mathcal{A}_{n,x,y} := \Big\{a_n d|\zeta_{n,x,y}|^{-1} \leq h(\xi_1, \xi_2) < a_n d'|\zeta_{n,x,y}|^{-1}\Big\}$$

and

$$\mathcal{B}_{n,x,y} := \Big\{a_n d|\zeta_{n,x,y}|^{-1} \leq -h(\xi_1, \xi_2) < a_n d'|\zeta_{n,x,y}|^{-1}\Big\}.$$

Observe that, due to (34) and to $\tilde{N}^*_n = o(n^{1-\frac{1}{\alpha_0}+\varepsilon})$, we have

$$\forall\gamma_0 > 0, \quad a_n^{-1} \sup_{x,y} |\zeta_{n,x,y}| \leq C a_n^{-1}(\tilde{N}^*_n)^2 \leq n^{-\frac{2}{\alpha_0\beta}+2\varepsilon+\gamma_0}, \tag{37}$$

for n large enough (and for some constant $C > 0$ depending on θ_i or on $\theta_{i,j}$). Now, combining this with Item (iii) of Assumption 1, we have

$$\begin{aligned}
\sum_{x,y:x\neq y} \mathbf{P}(\mathcal{A}_{n,x,y}|\tilde{\mathcal{F}})\mathbf{1}_{\{\zeta_{n,x,y}>0\}} &= c_0(d^{-\beta} - d'^{-\beta})a_n^{-\beta} \sum_{x,y\in\mathbb{Z}^{d_0}:x\neq y} |\zeta_{n,x,y}|^\beta \frac{\operatorname{sgn}(\zeta_{n,x,y}) + 1}{2} \\
&\quad \times \left(1 + O\left(\sup_{z>n^{\frac{2}{\alpha_0\beta}-2\varepsilon-\gamma_0}} |L_0(z) - c_0|\right)\right) + o(1) \\
&= c_0(d^{-\beta} - d'^{-\beta})\frac{\tilde{G}^+_n + \tilde{G}^-_n}{2} + o(1),
\end{aligned}$$

since $\varepsilon < 1/(\alpha_0\beta)$ and since, for n large enough,

$$\sum_{x\in\mathbb{Z}^{d_0}} |\zeta_{n,x,y}|^\beta \le n^{\frac{1}{\alpha_0}+\varepsilon} n^{2\beta-\frac{2\beta}{\alpha_0}+2\varepsilon\beta} = o(a_n^\beta),$$

since $\varepsilon < 1/((1+2\beta)\alpha_0)$. Analogously, we have

$$\begin{aligned}
\sum_{x,y:x\neq y} \mathbf{P}(\mathcal{B}_{n,x,y}|\tilde{\mathcal{F}})\mathbf{1}_{\{\zeta_{n,x,y}<0\}} &= c_1(d^{-\beta}-d'^{-\beta})a_n^{-\beta} \sum_{x,y\in\mathbb{Z}^{d_0}:x\neq y} |\zeta_{n,x,y}|^\beta \, \frac{1-\mathrm{sgn}(\zeta_{n,x,y})}{2} \\
&\quad \times \left(1+O\left(\sup_{z>n^{\frac{2}{\alpha_0\beta}-2\varepsilon-\gamma_0}} |L_1(z)-c_1|\right)\right)+o(1) \\
&= c_1(d^{-\beta}-d'^{-\beta})\frac{\tilde{G}_n^+-\tilde{G}_n^-}{2}+o(1),
\end{aligned}$$

We obtain (35) for $Q=[d,d')\subset(0,+\infty)$ using (1), (2) and the definition of $\tilde{G}_n^\pm$ and of $\tilde{G}^\pm$. The proof of (35) for $Q=(-d',-d]\subset(-\infty,0)$ follows the same scheme.

Now let us prove (36). Let $K\ge 1$ and let R be a union of K pairwise disjoint intervals $Q_1,...,Q_K$ with $Q_i:=(d_i,d_i']\subset(0,+\infty)$ or $Q_i:=[-d_i',-d_i)\subset(-\infty,0)$. We write $P_n^{\tilde{\omega}}$ for the Poisson distribution of intensity $\eta_n^{\tilde{\omega}}(R):=\mathbf{E}[\mathcal{N}_n(R)|\tilde{\mathcal{F}}](\tilde{\omega})$. On $\tilde{\Omega}_0$, due to (35), we have

$$|e^{-\eta_{\tilde{\omega}}(R)}-P_n^{\tilde{\omega}}(0)|=o(1).$$

Hence, to prove (36), we just have to prove

$$|\mathbf{P}(\mathcal{N}_n(R)=0|\tilde{\mathcal{F}})-P_n(0)|=o(1). \tag{38}$$

Following [1] and [10], we introduce the following notations. For every $x,y\in\mathbb{Z}^{d_0}$ such that $x\neq y$, we define the random variables

$$I_{x,y}=\sum_{i=1}^K \mathbf{1}_{\{h(\xi_x,\xi_y)\in a_n(\zeta_{n,x,y})^{-1}Q_i\}}.$$

Observe that

$$\mathcal{N}_n(R)=\sum_{x,y\in\mathbb{Z}^{d_0}:x\neq y} I_{x,y} \quad\text{and so}\quad \eta_n(R)=\sum_{x,y\in\mathbb{Z}^{d_0}:x\neq y}\mathbf{E}[I_{x,y}|\tilde{\mathcal{F}}]. \tag{39}$$

We will use the following lemma, whose proof is postponed until the end of this paragraph:

Lemma 14. *We have*

$$|\mathbf{P}(\mathcal{N}_n(R)=0|\tilde{\mathcal{F}})-P_n(0)|\le \min(1,(\eta_n(R))^{-1})(A_1+A_2),$$

with

$$A_1:=\sum_{(x,y)\in M}\mathbf{E}[I_{x,y}|\tilde{\mathcal{F}}]\mathbf{E}\left[I_{x,y}+\sum_{(x',y')\in M_{x,y}^{(1)}} I_{x',y'}\,\middle|\,\tilde{\mathcal{F}}\right],$$

$$A_2:=\sum_{(x,y)\in M}\mathbf{E}\left[I_{x,y}\left(I_{x,y}+\sum_{(x',y')\in M_{x,y}^{(1)}} I_{x',y'}\right)\middle|\,\tilde{\mathcal{F}}\right],$$

and with the notation $M_{x,y}^{(k)}:=\{(x',y')\in M:\#\{x',y'\}\cap\{x,y\}=k\}$ *and* $M:=\{(x,y)\in\mathbb{Z}^{2d_0}:x\neq y\}$.

To conclude, we have to prove that A_1 and A_2 converge to 0 as n goes to infinity.

We set $d := \min_i d_i$.

For A_1, using (1), (2) and the definition of $I_{x,y}$, we observe that, for $\gamma_0 > 0$ small enough, we have

$$\begin{aligned} A_1 &\leq 4 \sum_{x,y\in\mathbb{Z}^{d_0}} \sum_{x'\in\mathbb{Z}^{d_0}} \mathbf{P}\left(da_n \left|\zeta_{n,x,y}\right|^{-1} \leq |h(\xi_x,\xi_y)| \Big| \tilde{\mathcal{F}}\right) \\ &\qquad \times \mathbf{P}\left(da_n \left|\zeta_{n,x,x'}\right|^{-1} \leq |h(\xi_x,\xi_{x'})| \Big| \tilde{\mathcal{F}}\right) \\ &\leq C d^{-2\beta} a_n^{-2\beta} (\|L_0\|_\infty + \|L_1\|_\infty)^2 \tilde{R}_n^3 (\tilde{N}_n^*)^{4\beta} \\ &\leq O(n^{-\frac{1}{\alpha_0}+(4\beta+3)\varepsilon+\gamma_0}) = o(1), \end{aligned}$$

using $\varepsilon(4\beta+3) < 1/\alpha_0$, (34) together with the definitions of $\tilde{R}_n$ and $\tilde{N}_n^*$ (with C some constant depending on θ_j and $\theta_{i,j}$).

Now let us study A_2. We have, for $\gamma_0 > 0$ small enough,

$$\begin{aligned} A_2 &\leq 4 \sum_{x,y,x'\in\mathbb{Z}^{d_0}} \mathbf{P}\left(da_n \left|\zeta_{n,x,y}\right|^{-1} \leq |h(\xi_x,\xi_y)|, da_n \left|\zeta_{n,x,x'}\right|^{-1} \leq |h(\xi_x,\xi_{x'})| \Big| \tilde{\mathcal{F}}\right) \\ &\leq 4C_0 \tilde{R}_n^3 a_n^{-2\gamma} (\tilde{N}_n^*)^{4\gamma} \\ &\leq O\left(n^{\frac{3}{\alpha_0}+(3+4\gamma)\varepsilon-\frac{4\gamma}{\alpha_0\beta}+\gamma_0}\right) = o(1), \end{aligned}$$

due to $(3+4\gamma)\varepsilon\alpha_0 < \frac{4\gamma}{\beta} - 3$ (recall that this is possible since $\gamma > 3\beta/4$) and where C_0 is a constant depending on on d, θ_j and $\theta_{i,j}$. ☐

Proof of Lemma 14. The proof of this lemma follows the line of arguments that can be found in [10]. Let f be defined on $\mathbb{N}$ by $f(0) = 0$ and

$$f(m) := e^{\eta_n(R)} \frac{(m-1)!}{(\eta_n(R))^m} P_n(\{0\}) P_n([m,+\infty)).$$

We will use the two following inequalities (see [1] p.400 and p.401)

$$\left| \mathbf{P}(\mathcal{N}_n(R) = 0|\tilde{\mathcal{F}}) - P_n(0) \right| \leq \left| \mathbf{E}\Big[\eta_n(R) f(\mathcal{N}_n(R)+1) - \mathcal{N}_n(R) f(\mathcal{N}_n(R)) \Big| \tilde{\mathcal{F}} \Big] \right| \tag{40}$$

and

$$\sup_m |f(m+1) - f(m)| \leq \min(1, (\eta_n(R))^{-1}). \tag{41}$$

Now we observe that, for every $(x,y) \in (\mathbb{Z}^{d_0})^2$ such that $x \neq y$, we have

$$\mathcal{N}_n(R) = \sum_{x',y'\in\mathbb{Z}^{d_0}: x'\neq y'} I_{x',y'} = I_{x,y} + \mathcal{N}_{n,x,y}^{(0)} + \mathcal{N}_{n,x,y}^{(1)}, \tag{42}$$

with $\mathcal{N}_{n,x,y}^{(i)} := \sum_{(x',y')\in M_{x,y}^{(i)}} I_{x',y'}$. Starting from (40) and using (39), we have

$$\left| \mathbf{P}(\mathcal{N}_n(R) = 0|\tilde{\mathcal{F}}) - P_n(0) \right| \leq A_1' + A_2',$$

with

$$A_1' := \left| \sum_{x,y\in\mathbb{Z}^{d_0}: x\neq y} \mathbf{E}[I_{x,y}|\tilde{\mathcal{F}}] \mathbf{E}\Big[f(\mathcal{N}_n(R)+1) - f(\mathcal{N}_{n,x,y}^{(0)}+1) \Big| \tilde{\mathcal{F}} \Big] \right|$$

and

$$A_2' := \left| \sum_{x,y\in\mathbb{Z}^{d_0}: x\neq y} \mathbf{E}\Big[I_{x,y} f(\mathcal{N}_n(R)) \Big| \tilde{\mathcal{F}} \Big] - \mathbf{E}[I_{x,y}|\tilde{\mathcal{F}}] \mathbf{E}\Big[f(\mathcal{N}_{n,x,y}^{(0)}+1) \Big| \tilde{\mathcal{F}} \Big] \right|.$$

Now, using (41) and (42), we obtain

$$\begin{aligned}\left|f(\mathcal{N}_n(R)+1)-f(\mathcal{N}^{(0)}_{n,x,y}+1)\right| &\leq \sup_{m\geq 0}|f(m+1)-f(m)|\times\left(\mathcal{N}_n(R)-\mathcal{N}^{(0)}_{n,x,y}\right)\\ &\leq \min(1,(\eta_n(R))^{-1})(I_{x,y}+\mathcal{N}^{(1)}_{n,x,y}) \qquad (43)\end{aligned}$$

and so $A_1' \leq \min(1,(\eta_n(R))^{-1})A_1$. Observe that, conditioned with respect to $\tilde{\mathcal{F}}$, $I_{x,y}$ and $\mathcal{N}^{(0)}_{n,x,y}$ are independent. Therefore

$$A_2' = \left|\sum_{x,y\in\mathbb{Z}^{d_0}:x\neq y}\mathbf{E}\left[I_{x,y}\{f(\mathcal{N}_n(R))-f(\mathcal{N}^{(0)}_{n,x,y}+1)\}\Big|\tilde{\mathcal{F}}\right]\right|.$$

Now, using (41) once again, we obtain

$$\begin{aligned}\left|f(\mathcal{N}_n(R))-f(\mathcal{N}^{(0)}_{n,x,y}+1)\right| &\leq \min(1,(\eta_n(R))^{-1})(\mathcal{N}_n(R)-\mathcal{N}^{(0)}_{n,x,y})\\ &\leq \min(1,(\eta_n(R))^{-1})(I_{x,y}+\mathcal{N}^{(1)}_{n,x,y})\end{aligned}$$

and so $A_2' \leq \min(1,(\eta_n(R))^{-1})A_2$, which completes the proof of the lemma. □

4.3. **Proof of the convergence of the finite dimensional distributions.** In this paragraph we will finish the proof of the convergence of the finite dimensional distributions. Similarly to the proof given in [10], we will use the convergence of the associated point process and the continuous mapping theorem. The approach is based on the following observation:

$$a_n^{-1}\sum_{x,y}\zeta_{n,x,y}h(\xi_x,\xi_y)=\int_{\mathbb{R}^*}w\,d\mathcal{N}_n(w).$$

However the functional is not continuous and we will have to do some truncation. This will be the purpose of the three following propositions.

Proposition 15. *Let $\delta>0$. For $\tilde{\mathbb{P}}$ almost every $\tilde\omega\in\tilde\Omega_0$, the sequence of random variables*

$$Z_n^{\tilde\omega} := a_n^{-1}\sum_{x,y}\zeta_{n,x,y}(\tilde\omega)h(\xi_x,\xi_y)\mathbf{1}_{\{a_n^{-1}|\zeta_{n,x,y}(\tilde\omega)h(\xi_x,\xi_y)|>\delta\}} = \int_{\mathbb{R}^*}w\mathbf{1}_{(\delta,+\infty)}(|w|)\,d\mathcal{N}_n^{\tilde\omega}(w)$$

converges in distribution to $\int_{\mathbb{R}^*}w\mathbf{1}_{(\delta,+\infty)}(|w|)\,d\mathcal{N}^{\tilde\omega}(w)$.

Proposition 16. *For every $\gamma_0>0$, we have*

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbf{P}\left(|T_n(\delta)|>\gamma_0|\tilde{\mathcal{F}}\right)=0 \quad \tilde{\mathbb{P}}-a.s.,$$

with

$$T_n(\delta) := a_n^{-1}\sum_{x,y}\zeta_{n,x,y}h(\xi_x,\xi_y)\mathbf{1}_{\{a_n^{-1}|\zeta_{n,x,y}h(\xi_x,\xi_y)|\leq\delta\}} \quad \text{if } \beta\leq 1$$

and

$$T_n(\delta) := a_n^{-1}\sum_{x,y}\zeta_{n,x,y}h(\xi_x,\xi_y)\mathbf{1}_{\{a_n^{-1}|\zeta_{n,x,y}h(\xi_x,\xi_y)|\leq\delta\}}+(c_0-c_1)\frac{\beta\delta^{1-\beta}}{\beta-1}\tilde G_n^- \quad \text{if } \beta>1.$$

Proposition 17 (see [22])**.** *Let $\mathcal{P}$ be a Poisson process on $\mathbb{R}^*$ with intensity admitting the density $z\mapsto\beta|z|^{-\beta-1}(a\mathbf{1}_{\{z>0\}}+b\mathbf{1}_{\{z<0\}})$.*

If $\beta<1$, then $\int_{\mathbb{R}^*\setminus[-\delta,\delta]}w\,d\mathcal{P}(w)$ converges in distribution, as δ goes to 0, to a stable random variable with characteristic function $\Phi_{a+b,a-b,\beta}$ with the notation of (4).

If $\beta=1$, then $\int_{\mathbb{R}^*\setminus[-\delta,\delta]} w\,d\mathcal{P}(w)-(a-b)\int_\delta^{+\infty}\frac{\sin x}{x^2}\,dx$ converges in distribution, as δ goes to 0, to a stable random variable with characteristic function $\Phi_{a+b,a-b,1}$, with the notation of (5).

If $\beta>1$, then $\int_{\mathbb{R}^*\setminus[-\delta,\delta]} w\,d\mathcal{P}(w)-(a-b)\frac{\beta\delta^{1-\beta}}{\beta-1}$ converges in distribution, as δ goes to 0, to a stable random variable with characteristic function $\Phi_{a+b,a-b,\beta}$ with the notation of (4).

The following corollary is a consequence of Propositions 13, 15, 16 and 17.

Corollary 18. *We have*

$$\lim_{n\to+\infty}\mathbf{E}[e^{ia_n^{-1}\sum_{x,y}\zeta_{n,x,y}(\tilde\omega)h(\xi_x,\xi_y)}|\tilde{\mathcal{F}}]=\Phi_{(c_0+c_1)\tilde G^+(\tilde\omega),(c_0-c_1)\tilde G^-(\tilde\omega),\beta}(1),$$

for $\tilde{\mathbb{P}}$-almost every $\tilde\omega$ in $\tilde\Omega$ and

$$\lim_{n\to+\infty}\mathbf{E}\left[e^{ia_n^{-1}\sum_{x,y}\zeta_{n,x,y}h(\xi_x,\xi_y)}\right]=\mathbf{E}\left[\Phi_{(c_0+c_1)\tilde G^+,(c_0-c_1)\tilde G^-,\beta}(1)\right].$$

Proof of Corollary 18. Observe first that due to the Lebesgue dominated convergence theorem it is enough to prove the first convergence. Let $\tilde\Omega_1$ be the subset of $\tilde\Omega_0$ on which the convergences of Propositions 15 and 16 hold and let $\tilde\omega\in\tilde\Omega_1$. To simplify notations, let us write

$$V_n:=a_n^{-1}\sum_{x,y}\zeta_{n,x,y}h(\xi_x,\xi_y)\quad\text{and}\quad W_n(\delta):=a_n^{-1}\sum_{x,y}\zeta_{n,x,y}h(\xi_x,\xi_y)\mathbf{1}_{\{a_n^{-1}|\zeta_{n,x,y}h(\xi_x,\xi_y)|>\delta\}}.$$

We set $\kappa:=0$ if $\beta\le 1$ and $\kappa:=(c_0-c_1)\frac{\beta}{\beta-1}$ if $\beta>1$ (recall that we assume $c_0=c_1$ if $\beta=1$). We also write $W_{\tilde\omega}(\delta):=\int_{\mathbb{R}\setminus[-\delta,\delta]} w\,d\mathcal{N}_{\tilde\omega}(w)$ (where $\mathcal{N}_{\tilde\omega}$ is the Poisson process of Proposition 13, which is defined on some probability space $(\Omega_{\tilde\omega},\mathcal{T}_{\tilde\omega},\mathrm{P}_{\tilde\omega})$ endowed with the expectation $\mathrm{E}_{\tilde\omega}$). Let $\epsilon>0$. Due to Propositions 16, 13 and 17, we consider $\delta>0$ and n_0 such that, for every $n\ge n_0$, we have

$$\mathbf{P}\left(|T_n(\delta)|>\frac{\epsilon}{6}\middle|\tilde{\mathcal{F}}\right)(\tilde\omega)<\frac{\epsilon}{6}\tag{44}$$

and such that

$$\left|\mathrm{E}_{\tilde\omega}\left[e^{i(W_{\tilde\omega}(\delta)-\kappa\delta^{1-\beta}\tilde G^-(\tilde\omega))}\right]-\Phi_{(c_0+c_1)\tilde G^+(\tilde\omega),(c_0-c_1)\tilde G^-(\tilde\omega),\beta}(1)\right|<\frac{\epsilon}{6}.\tag{45}$$

Due to Proposition 15, we consider $n_1\ge n_0$ such that, for every $n\ge n_1$, we have

$$\left|\mathbf{E}[e^{iW_n(\delta)}|\tilde{\mathcal{F}}](\tilde\omega)-\mathrm{E}_{\tilde\omega}[e^{iW_{\tilde\omega}(\delta)}]\right|<\frac{\epsilon}{6}.\tag{46}$$

Now, let $n_2\ge n_1$ such that, for every $n\ge n_2$, we have

$$\left|e^{i\kappa\delta^{1-\beta}\tilde G^-(\tilde\omega)}-e^{i\kappa\delta^{1-\beta}\tilde G_n^-(\tilde\omega)}\right|<\frac{\epsilon}{6}.\tag{47}$$

For $n\ge n_2$, we have

$$\begin{aligned}
&\left|\mathbf{E}[e^{iV_n}|\tilde{\mathcal{F}}](\tilde\omega)-\Phi_{(c_0+c_1)\tilde G^+(\tilde\omega),(c_0-c_1)\tilde G^-(\tilde\omega),\beta}(1)\right|\\
\le\;&\frac{\epsilon}{6}+\left|\mathbf{E}[e^{iV_n}|\tilde{\mathcal{F}}](\tilde\omega)-\mathrm{E}_{\tilde\omega}[e^{i(W_{\tilde\omega}(\delta)-\kappa\delta^{1-\beta}\tilde G^-(\tilde\omega))}]\right|\quad\text{due to (45)}\\
\le\;&\frac{\epsilon}{6}+\left|\mathbf{E}[e^{i(V_n+\kappa\delta^{1-\beta}\tilde G^-)}|\tilde{\mathcal{F}}](\tilde\omega)-\mathrm{E}_{\tilde\omega}[e^{i(W_{\tilde\omega}(\delta)}]\right|\\
\le\;&\frac{2\epsilon}{6}+\left|\mathbf{E}[e^{i(V_n+\kappa\delta^{1-\beta}\tilde G_n^-)}|\tilde{\mathcal{F}}](\tilde\omega)-\mathrm{E}_{\tilde\omega}[e^{i(W_{\tilde\omega}(\delta)}]\right|\quad\text{due to (47)}\\
\le\;&\frac{2\epsilon}{6}+\left|\mathbf{E}[e^{i(W_n(\delta)+T_n(\delta))}|\tilde{\mathcal{F}}](\tilde\omega)-\mathrm{E}_{\tilde\omega}[e^{i(W_{\tilde\omega}(\delta)}]\right|\\
\le\;&\frac{3\epsilon}{6}+\left|\mathbf{E}[e^{i(W_n(\delta)+T_n(\delta))}-e^{iW_n(\delta)}|\tilde{\mathcal{F}}](\tilde\omega)\right|\quad\text{due to (46)}\\
\le\;&\frac{4\epsilon}{6}+2\mathbf{P}\left(|T_n(\delta)|>\frac{\epsilon}{6}\middle|\tilde{\mathcal{F}}\right)(\tilde\omega)\le\epsilon\quad\text{due to (44).}
\end{aligned}$$

□

Proof of the convergence of finite distributions in Theorems 3, 5 and 6. Admitting Propositions 15, 16 and 17 for the moment, let us end the proof of the convergence of the finite distributions. Due to Corollary 18, we have

$$\begin{aligned}\lim_{n\to+\infty}\mathbf{E}[e^{ia_n^{-1}\sum_{x,y}\zeta_{n,x,y}h(\xi_x,\xi_y)}] &= \mathbf{E}\left[\Phi_{(c_0+c_1)\tilde{G}^+,(c_0-c_1)\tilde{G}^-,\beta}(1)\right]\\ &= \mathbb{E}\left[\exp\left(-\int_0^{+\infty}\frac{\sin t}{t^\beta}\,dt\left[(c_0+c_1)G^+ - i(c_0-c_1)G^-\tan\frac{\pi\beta}{2}\right]\right)\right].\end{aligned}$$

When $\alpha_0 = 1$, with the use of (10) and (14) , we obtain

$$\begin{aligned}&\lim_{n\to+\infty}\mathbb{E}\left[e^{ia_n^{-1}\sum_{j=1}^m\theta_j(U_{\lfloor nt_j\rfloor}-U_{\lfloor nt_{j-1}\rfloor})}\right]\\ =\ &\exp\left(-K_\beta^2\sum_{i=1}^m(t_i^2-t_{i-1}^2)|\theta_i|^\beta\int_0^{+\infty}\frac{\sin t}{t^\beta}\,dt\left[(c_0+c_1)-i(c_0-c_1)\operatorname{sgn}(\theta_i)\tan\frac{\pi\beta}{2}\right]\right)\\ =\ &\prod_{j=1}^m\Phi_{(c_0+c_1)K_\beta^2(t_i^2-t_{i-1}^2),(c_0-c_1)K_\beta^2(t_i^2-t_{i-1}^2),\beta}(\theta_j)\end{aligned}$$

This gives the convergence of the finite distributions in Theorems 3 and 5.

When $\alpha_0 > 1$, due to Lemma 12, we obtain

$$\lim_{n\to+\infty}\mathbb{E}\left[e^{i\sum_{j=1}^m\theta_j a_n^{-1}U_{\lfloor nt_j\rfloor}}\right] = \mathbb{E}\left[\Phi_{(c_0+c_1)G^+,(c_0-c_1)G^-,\beta}(1)\right], \tag{48}$$

with $G^\pm = \int_{\mathbb{R}^2}|\sum_{i=1}^m\theta_i\mathcal{L}_{t_i}(x)\mathcal{L}_{t_i}(y)|_\pm^\beta\,dxdy$. Let us recall that the right hand side of (48) corresponds to the characteristic function of $\sum_{i=1}^m\theta_i\int_{\mathbb{R}^2}\mathcal{L}_{t_i}(x)\mathcal{L}_{t_i}(y)\,dZ_{x,y}$ evaluated at one (see for example [18] and Appendix A). □

Proof of Proposition 15. To simplify notations we also write $\mathrm{P}_{\tilde\omega}$ for $\mathbf{P}(\cdot|\tilde{\mathcal{F}})(\tilde\omega)$ and $\mathrm{E}_{\tilde\omega}$ for $\mathbf{E}[\cdot|\tilde{\mathcal{F}}](\tilde\omega)$.

We proceed in four steps:

1) We first use the continuous mapping theorem (see [21] p.151) to prove that for $\tilde{\mathbb{P}}$-almost all $\tilde\omega$ one has

$$\int_{(-M,-\delta)\cup(\delta,M)} z d\mathcal{N}_n^{\tilde\omega}(dz) \xrightarrow{\mathcal{L}} \int_{(-M,-\delta)\cup(\delta,M)} z d\mathcal{N}^{\tilde\omega}(dz). \tag{49}$$

The Poisson process $\tilde{\mathcal{N}}_{\tilde\omega}$ has $\tilde{\mathbb{P}}$-almost surely only a finite number of points in the interval $(-M,-\delta)\cup(\delta,M)$. Moreover, one has $\tilde{\mathbb{P}}$-almost surely that each of those points only carries the mass one, since the Poisson process $\tilde{\mathcal{N}}_{\tilde\omega}$ is simple. Now, let μ be a point measure with only a finite number of points with mass one in $(-M,-\delta)\cup(\delta,M)$ and let $(\mu_n)_{n\in\mathbb{N}}$ be some sequence of point measures which converges toward μ with respect to the vague topology on $\mathbb{R}^*$. Let $\{x_1,...,x_p\}$ be the support of μ intersected with $(-M,-\delta)\cup(\delta,M)$. According to [20] (see Lemma I.14) there exists some large $N\in\mathbb{N}$ such that for all $n\geq N$ the support of μ_n intersected with $(-M,-\delta)\cup(\delta,M)$ in exactly p point $x_1^{(n)},...,x_p^{(n)}$ such that

$$\lim_{n\to\infty}x_i^{(n)} = x_i \quad \text{for all } i=1,...,p.$$

It then follows that

$$\lim_{n\to\infty}\int_{(-M,-\delta)\cup(\delta,M)} z\mu_n(dz) = \lim_{n\to\infty}\sum_{i=1}^p x_i^{(n)} = \sum_{i=1}^p x_i = \int_{(-M,-\delta)\cup(\delta,M)} z\mu(dz).$$

2) We now prove that for $\tilde{\mathbb{P}}$-almost all $\tilde\omega$ one has

$$\int_{(-\infty,-M)\cup(M,\infty)} z d\mathcal{N}^{\tilde\omega}(dz) \xrightarrow{\mathrm{P}_{\tilde\omega}} 0 \quad \text{as } M\to\infty. \tag{50}$$

This follows from the following equality which holds for $\tilde{\mathbb{P}}$-almost all $\tilde\omega$

$$\mathrm{E}_{\tilde\omega}\left[\exp\left(it\int_M^\infty z\mathcal{N}^{\tilde\omega}(dz)\right)\right] = \exp\left((c_0+c_1)\tilde{G}^+\int_M^\infty \beta\frac{\cos(tx)-1}{x^{\beta+1}}dx + i(c_0-c_1)\tilde{G}^-\int_M^\infty \beta\frac{\sin(tx)}{x^{\beta+1}}dx\right)$$

and from the fact that one has

$$\left|(c_0+c_1)\tilde{G}^+\int_M^\infty \beta\frac{\cos(tx)-1}{x^{\beta+1}}dx + i(c_0-c_1)\tilde{G}^-\int_M^\infty \beta\frac{\sin(tx)}{x^{\beta+1}}dx\right| \le 2M^{-\beta}\Big((c_0+c_1)(|\tilde{G}^+|+|\tilde{G}^-|)\Big).$$

This yields

$$\mathrm{E}_{\tilde\omega}\left[\exp\left(it\int_M^\infty z\mathcal{N}^{\tilde\omega}(dz)\right)\right] \longrightarrow 1 \quad \text{for } \tilde{\mathbb{P}} \text{ almost all } \tilde\omega \text{ as } M\to\infty.$$

The convergence in probability follows from the convergence in law of $\int_M^\infty z\mathcal{N}^{\tilde\omega}(dz)$ toward zero. The other part $\int_{-\infty}^{-M} z\mathcal{N}^{\tilde\omega}(dz)$ is treated in the same way.

3) We now prove that for $\tilde{\mathbb{P}}$-almost all $\tilde\omega$ we have

$$\sup_{n\in\mathbb{N}} \mathrm{P}_{\tilde\omega}\left(\int_{(-\infty,-M)\cup(M,\infty)} z\mathcal{N}_n^{\tilde\omega}(dz)\neq 0\right) \longrightarrow 0 \quad \text{as } M\to\infty. \tag{51}$$

For this first remember that

$$\int_{(-\infty,-M)\cup(M,\infty)} z\mathcal{N}_n^{\tilde\omega}(dz) = \sum_{x,y\in\mathbb{Z}} a_n^{-1}\zeta_{n,x,y}h(\xi_x,\xi_y)\mathbf{1}_{\{|a_n^{-1}\zeta_{n,x,y}h(\xi_x,\xi_y)|>M\}}.$$

Thus this implies

$$\begin{aligned}
\mathrm{P}_{\tilde\omega}\left(\int_{\{|z|>M\}} z\mathcal{N}_n^{\tilde\omega}(dz)\neq 0\right) &\le \mathrm{P}_{\tilde\omega}\Big(\exists x,y\in\mathbb{Z}:\ |a_n^{-1}\zeta_{n,x,y}h(\xi_x,\xi_y)|>M\Big)\\
&\le \sum_{x,y\in\mathbb{Z}} \mathrm{P}_{\tilde\omega}\Big(|h(\xi_x,\xi_y)|>Ma_n|\zeta_{n,x,y}|^{-1}\Big)\\
&\le \sum_{x,y\in\mathbb{Z}} C\Big(Ma_n|\zeta_{n,x,y}|^{-1}\Big)^{-\beta}\\
&\le CM^{-\beta}a_n^{-\beta}\sum_{x,y\in\mathbb{Z}}|\zeta_{n,x,y}|^\beta = CM^{-\beta}G_n^+ \longrightarrow 0 \quad \text{as } M\to\infty,
\end{aligned}$$

since $\mathbb{P}$-almost surely we have $G_n^+\to G^+$ as $n\to\infty$.

4) We now use the previous findings to conclude. We consider an $\tilde\omega$ which satisfies all the requirements from points (1) to (3) of this proof. For some given $t\in\mathbb{R}$ and $\epsilon>0$ we use (51) to find some $M>0$ such that

$$\sup_{n\in\mathbb{N}} \mathrm{P}_{\tilde\omega}\left(\int_{(-\infty,-M)\cup(M,\infty)} z\mathcal{N}_n^{\tilde\omega}(dz)\neq 0\right) \le \epsilon/8$$

By (50) we can assume without loss of generality that the M also satisfies

$$\mathrm{P}_{\tilde\omega}\left(t\left|\int_{(-\infty,-M)\cup(M,\infty)} z d\mathcal{N}^{\tilde\omega}(dz)\right| \ge \epsilon/4\right) \le \epsilon/8.$$

Moreover, according to (49) we can find some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\left|\mathrm{E}_{\tilde{\omega}}\left[\exp\left(it\int_{(-M,-\delta)\cup(\delta,M)} z\mathcal{N}_n^{\tilde{\omega}}(dz)\right)\right] - \mathrm{E}_{\tilde{\omega}}\left[\exp\left(it\int_{(-M,-\delta)\cup(\delta,M)} z\mathcal{N}^{\tilde{\omega}}(dz)\right)\right]\right| \leq \epsilon/4.$$

It now follows that

$$\begin{aligned}
&\left|\mathrm{E}_{\tilde{\omega}}\left[\exp\left(it\int_{(-\infty,-\delta)\cup(\delta,\infty)} z\mathcal{N}_n^{\tilde{\omega}}(dz)\right)\right] - \mathrm{E}_{\tilde{\omega}}\left[\exp\left(it\int_{(-\infty,-\delta)\cup(\delta,\infty)} z\mathcal{N}^{\tilde{\omega}}(dz)\right)\right]\right| \\
= &\left|\mathrm{E}_{\tilde{\omega}}\left[\exp\left(it\int_{(-M,-\delta)\cup(\delta,M)} z\mathcal{N}_n^{\tilde{\omega}}(dz)\right)\left(1+\exp\left(it\int_{(-\infty,-M)\cup(M,\infty)} z\mathcal{N}_n^{\tilde{\omega}}(dz)\right)-1\right)\right]\right. \\
&\left.-\mathrm{E}_{\tilde{\omega}}\left[\exp\left(it\int_{(-M,-\delta)\cup(\delta,M)} z\mathcal{N}^{\tilde{\omega}}(dz)\right)\left(1+\exp\left(it\int_{(-\infty,-M)\cup(M,\infty)} z\mathcal{N}^{\tilde{\omega}}(dz)\right)-1\right)\right]\right| \\
\leq &\left|\mathrm{E}_{\tilde{\omega}}\left[\exp\left(it\int_{(-M,-\delta)\cup(\delta,M)} z\mathcal{N}_n^{\tilde{\omega}}(dz)\right)\right] - \mathrm{E}_{\tilde{\omega}}\left[\exp\left(it\int_{(-M,-\delta)\cup(\delta,M)} z\mathcal{N}^{\tilde{\omega}}(dz)\right)\right]\right| \\
&+2\mathrm{P}_{\tilde{\omega}}\left(\int_{(-\infty,-M)\cup(M,\infty)} z\mathcal{N}_n^{\tilde{\omega}}(dz) \neq 0\right) + 2\mathrm{P}_{\tilde{\omega}}\left(t\left|\int_{(-\infty,-M)\cup(M,\infty)} zd\mathcal{N}^{\tilde{\omega}}(dz)\right| \geq \epsilon/4\right) + \frac{\epsilon}{4}.
\end{aligned}$$

Since the right side is equal to ϵ this finishes the proof of the proposition. □

Proof of Proposition 16.

- When $\beta < 1$, we just prove that $\lim_{\delta\to 0}\limsup_{n\to\infty} \mathbf{E}[|T_n(\delta)||\tilde{\mathcal{F}}] = 0$. Due to Item (iii) of Assumption 1, we have

$$\begin{aligned}
\mathbf{E}[|T_n(\delta)||\tilde{\mathcal{F}}] &\leq \sum_{x,y} \mathbf{E}\left[a_n^{-1}|\zeta_{n,x,y}h(\xi_x,\xi_y)|\mathbf{1}_{\{a_n^{-1}|\zeta_{n,x,y}h(\xi_x,\xi_y)|\leq\delta\}}\middle|\tilde{\mathcal{F}}\right] \\
&\leq \sum_{x,y}\int_0^\delta \mathbf{P}\left(\delta \geq a_n^{-1}|h(\xi_x,\xi_y)\zeta_{n,x,y}| > z\middle|\tilde{\mathcal{F}}\right)dz \\
&\leq \sum_{x,y}\int_0^\delta \mathbf{P}\left(|h(\xi_x,\xi_y)\zeta_{n,x,y}| > a_n z\middle|\tilde{\mathcal{F}}\right)dz \\
&\leq (\|L_0\|_\infty + \|L_1\|_\infty)\sum_{x,y}\int_0^\delta a_n^{-\beta}z^{-\beta}(\zeta_{n,x,y})^\beta\,dz \\
&\leq (\|L_0\|_\infty + \|L_1\|_\infty)\sum_{x,y}\frac{a_n^{-\beta}\delta^{1-\beta}}{1-\beta}(\zeta_{n,x,y})^\beta \\
&\leq (\|L_0\|_\infty + \|L_1\|_\infty)\frac{\delta^{1-\beta}}{1-\beta}\tilde{G}_n^+.
\end{aligned}$$

  So $\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbf{E}[|T_n(\delta)||\tilde{\mathcal{F}}] \leq \lim_{\delta\to 0}(\|L_0\|_\infty + \|L_1\|_\infty)\frac{\delta^{1-\beta}}{1-\beta}\tilde{G}^+ = 0$, since $\beta < 1$.
- Assume here that $\beta \in (1,2)$. Observe that, due to Item (v) of Assumption 1, we have

$$\begin{aligned}
&\mathbb{E}\big[h(\xi_1,\xi_2)\mathbf{1}_{\{|h(\xi_1,\xi_2)|\le M\}}\big]\\
=\ & -\mathbb{E}\left[h(\xi_1,\xi_2)\mathbf{1}_{\{|h(\xi_1,\xi_2)|>M\}}\right]\\
=\ & \int_0^{+\infty}\mathbb{P}\Big(h(\xi_1,\xi_2)<-\max(z,M)\Big)\,dz-\int_0^{+\infty}\mathbb{P}\Big(h(\xi_1,\xi_2)>\max(z,M)\Big)\,dz\\
=\ & M\Big(\mathbb{P}(h(\xi_1,\xi_2)<-M)-\mathbb{P}(h(\xi_1,\xi_2)>M)\Big)+\int_M^{+\infty}\mathbb{P}(h(\xi_1,\xi_2)<-z)\,dz\\
&\qquad -\int_M^{+\infty}\mathbb{P}(h(\xi_1,\xi_2)>z)\,dz.
\end{aligned}$$

But, due to Item (iii) of Assumption 1, as x goes to infinity, we have

$$\mathbb{P}(h(\xi_1,\xi_2)>x)=c_0x^{-\beta}+o(x^{-\beta}),\quad \mathbb{P}(h(\xi_1,\xi_2)<-x)=c_1x^{-\beta}+o(x^{-\beta}),$$

$$\int_x^{+\infty}\mathbb{P}(h(\xi_1,\xi_2)>z)\,dz=c_0\frac{x^{1-\beta}}{\beta-1}+o(x^{1-\beta}),$$

$$\int_x^{+\infty}\mathbb{P}(h(\xi_1,\xi_2)<-z)\,dz=c_1\frac{x^{1-\beta}}{\beta-1}+o(x^{1-\beta})$$

and

$$\forall x>0,\quad \int_x^{+\infty}\Big(\mathbb{P}\big(h(\xi_1,\xi_2)>z\big)+\mathbb{P}\big(h(\xi_1,\xi_2)<-z\big)\Big)\,dz\le(\|L_0\|_\infty+\|L_1\|_\infty)\frac{x^{1-\beta}}{\beta-1}.$$

Therefore, we obtain

$$\mathbb{E}\left[h(\xi_1,\xi_2)\mathbf{1}_{\{|h(\xi_1,\xi_2)|\le M\}}\right]=M^{1-\beta}\left(\frac{\beta}{\beta-1}(c_1-c_0)+\epsilon_M\right), \tag{52}$$

where $\lim_{M\to+\infty}\epsilon_M=0$ and $\sup_{M>0}\epsilon_M<\infty$.

- When $\beta=1$, due to Item (vii) of Assumption 1, we have $c_0=c_1$ and (52) holds also true.
- Assume now that $\beta\in[1,2)$. We will prove that $\lim_{\delta\to0}\limsup_{n\to\infty}\mathbf{E}[(T_n(\delta))^2|\tilde{\mathcal{F}}]=0$. We have

$$\mathbf{E}[(T_n(\delta))^2|\tilde{\mathcal{F}}]=\sum_{x,y,x',y'\in\mathbb{Z}^{d_0}}\mathbf{E}[T_{n,x,y}T_{n,x',y'}|\tilde{\mathcal{F}}],$$

with

$$T_{n,x,y}:=a_n^{-1}h(\xi_x,\xi_y)\zeta_{n,x,y}\mathbf{1}_{\{|h(\xi_x,\xi_y)\zeta_{n,x,y}|\le a_n\delta\}}+a_n^{-\beta}(c_0-c_1)\frac{\beta\delta^{1-\beta}}{\beta-1}|\zeta_{n,x,y}|_-^\beta$$

(recall that $c_0=c_1$ when $\beta=1$).

  - Contribution of (x,y,x',y') such that $\{x,y\}\cap\{x',y'\}=\emptyset$.
    We set E_1 for the set of such (x,y,x',y'). Let $(x,y,x',y')\in E_1$. Since $h(\xi_x,\xi_y)$ and $h(\xi_{x'},\xi_{y'})$ are independent conditionally to $\tilde{\mathcal{F}}$, we have

$$\mathbf{E}[T_{n,x,y}T_{n,x',y'}|\tilde{\mathcal{F}}]=\mathbf{E}[T_{n,x,y}|\tilde{\mathcal{F}}]\mathbf{E}[T_{n,x',y'}|\tilde{\mathcal{F}}].$$

    Now, due to (52), we have

$$\left|\sum_{x,y\in\mathbb{Z}^{d_0}}\mathbf{E}[T_{n,x,y}|\tilde{\mathcal{F}}]\right|\le\delta^{1-\beta}\sum_{x,y\in\mathbb{Z}^{d_0}}a_n^{-\beta}|\zeta_{n,x,y}|_+^\beta\epsilon_{a_n\delta|\zeta_{n,x,y}|^{-1}}.$$

    Now, due to (37), for every $\gamma_0>0$, if n is large enough, we have

$$a_n^{-1}\sup_{x,y\in\mathbb{Z}^{d_0}}|\zeta_{n,x,y}|\le n^{-\frac{2}{\alpha_0\beta}+2\varepsilon+\gamma_0}.$$

Combining this with $\lim_{n\to+\infty}\tilde G_n^+=\tilde G^+$ and with $\lim_{M\to+\infty}\epsilon_M=0$, we obtain

$$\limsup_{n\to+\infty}\sum_{x,y\in\mathbb{Z}^{d_0}}\mathbf{E}[T_{n,x,y}|\tilde{\mathcal{F}}]=0, \tag{53}$$

since $\beta\epsilon<1/\alpha_0$. This implies

$$\forall\delta>0,\quad \limsup_{n\to+\infty}\sum_{(x,y,x',y')\in E_1}\mathbf{E}[T_{n,x,y}T_{n,x',y'}|\tilde{\mathcal{F}}]=0.$$

– Contribution of (x,y,x',y') such that $\{x,y\}=\{x',y'\}$.
Let us write E_2 for the set of such (x,y,x',y'). Observe that

$$\sum_{(x,y,x',y')\in E_2}\mathbf{E}[T_{n,x,y}T_{n,x',y'}|\tilde{\mathcal{F}}]\le 2\sum_{x,y\in\mathbb{Z}^{d_0}}\mathbf{E}[T^2_{n,x,y}|\tilde{\mathcal{F}}].$$

First, using Item (iii) of Assumption 1, we notice that

$$\begin{aligned}
&a_n^{-2}\sum_{x,y\in\mathbb{Z}^{d_0}}\mathbf{E}\Big[(h(\xi_1,\xi_2)\zeta_{n,x,y})^2\mathbf{1}_{\{|h(\xi_1,\xi_2)\zeta_{n,x,y}|\le a_n\delta\}}\Big|\tilde{\mathcal{F}}\Big]\\
&=\sum_{x,y\in\mathbb{Z}^{d_0}}\int_0^{\delta^2}\mathbb{P}\Big(\sqrt{z}<a_n^{-1}|h(\xi_1,\xi_2)\zeta_{n,x,y}|<\delta\Big|\tilde{\mathcal{F}}\Big)\,dz\\
&\le\sum_{x,y\in\mathbb{Z}^{d_0}}\int_0^{\delta^2}\mathbb{P}\Big(\sqrt{z}<a_n^{-1}|h(\xi_1,\xi_2)\zeta_{n,x,y}|\Big|\tilde{\mathcal{F}}\Big)\,dz\\
&\le(\|L_0\|_\infty+\|L_1\|_\infty)\sum_{x,y\in\mathbb{Z}^{d_0}}\int_0^{\delta^2}a_n^{-\beta}z^{-\frac{\beta}{2}}|\zeta_{n,x,y}|^\beta\,dz\\
&\le(\|L_0\|_\infty+\|L_1\|_\infty)a_n^{-\beta}\sum_{x,y\in\mathbb{Z}^{d_0}}|\zeta_{n,x,y}|^\beta\frac{\delta^{2\left(1-\frac{\beta}{2}\right)}}{1-\frac{\beta}{2}}\\
&\le(\|L_0\|_\infty+\|L_1\|_\infty)\tilde G_n^+\frac{\delta^{2-\beta}}{1-\frac{\beta}{2}}.
\end{aligned}$$

Therefore

$$\lim_{\delta\to0}\limsup_{n\to+\infty}a_n^{-2}\sum_{x,y\in\mathbb{Z}^{d_0}}\mathbf{E}\Big[(h(\xi_1,\xi_2)\zeta_{n,x,y})^2\mathbf{1}_{\{|h(\xi_1,\xi_2)\zeta_{n,x,y}|\le a_n\delta\}}\Big|\tilde{\mathcal{F}}\Big]=0. \tag{54}$$

Second, using (34) and the definition of $\tilde N_n^*$ and $\tilde R_n$, for every $\gamma_0>0$, for n large enough, we have

$$\begin{aligned}
a_n^{-2\beta}\left|\sum_{x,y\in\mathbb{Z}^{d_0}}\left((c_0-c_1)^2\frac{\beta^2\delta^{2-2\beta}}{(\beta-1)^2}|\zeta_{n,x,y}|_-^{2\beta}\right)\right| &\le (c_0-c_1)^2\frac{\beta^2\delta^{2-2\beta}}{(\beta-1)^2}a_n^{-2\beta}\tilde R_n^2(\tilde N_n^*)^{4\beta}\\
&\le n^{-\frac{2}{\alpha_0}+2\epsilon+4\beta\epsilon+\gamma_0}\delta^{2-2\beta}.
\end{aligned}$$

So, since $\epsilon>0$ satisfies $(3+4\beta)\epsilon<\frac{1}{\alpha_0}$ we have that

$$\lim_{\delta\to0}\limsup_{n\to+\infty}a_n^{-2\beta}\sum_{x,y\in\mathbb{Z}^{d_0}}\left((c_0-c_1)^2\frac{\beta^2\delta^{2-2\beta}}{(\beta-1)^2}|\zeta_{n,x,y}|_-^{2\beta}\right)=0. \tag{55}$$

Finally this shows

$$\lim_{\delta\to 0}\limsup_{n\to+\infty}\sum_{(x,y,x',y')\in E_2}\mathbf{E}[T_{n,x,y}T_{n,x',y'}|\tilde{\mathcal{F}}]=0.$$

– Contribution of (x,y,x',y') such that $\#(\{x,y\}\cap\{x',y'\})=1$.
Let us write E_3 for the set of such (x,y,x',y'). Observe that we have

$$\sum_{(x,y,x',y')\in E_3}\mathbf{E}[T_{n,x,y}T_{n,x',y'}|\tilde{\mathcal{F}}]=4\sum_{x,y,z:x\neq y,x\neq z,y\neq z}\mathbb{E}\left[T_{n,x,y}T_{n,x,z}|\tilde{\mathcal{F}}\right]$$

* Assume that $1\le\beta<4/3$. We set $U_{n,x,y}:=a_n^{-1}h(\xi_x,\xi_y)\zeta_{n,x,y}\mathbf{1}_{\{|h(\xi_x,\xi_y)\zeta_{n,x,y}|\le a_n\delta\}}$. Observe that

$$T_{n,x,y}=U_{n,x,y}+a_n^{-\beta}(c_0-c_1)\frac{\beta\delta^{1-\beta}}{\beta-1}|\zeta_{n,x,y}|_-^{\beta}\tag{56}$$

(recall that we assume $c_0=c_1$ if $\beta=1$) and that, due to (52),

$$\mathbb{E}[U_{n,x,y}|\tilde{\mathcal{F}}]=a_n^{-\beta}\delta^{1-\beta}|\zeta_{n,x,y}|_-^{\beta}\left[(c_1-c_0)\frac{\beta}{\beta-1}+\epsilon_{a_n\delta|\zeta_{n,x,y}|^{-1}}\right].\tag{57}$$

Now, (37) ensures that

$$\lim_{n\to+\infty}\sup_{x,y}\epsilon_{a_n\delta|\zeta_{n,x,y}|^{-1}}=0.\tag{58}$$

Moreover, we observe that, due to (34) and to the definition of $\tilde{N}_n^*$ and of $\tilde{R}_n$, we have, for every $\gamma_0>0$ and every n large enough,

$$\begin{aligned}\sum_{x,y,z\in\mathbb{Z}^{d_0}}a_n^{-2\beta}|\zeta_{n,x,y}|^{\beta}|\zeta_{n,x,z}|^{\beta} &\le \tilde{R}_n^3a_n^{-2\beta}\left(\tilde{N}_n^*\right)^{4\beta}\\ &\le n^{-\frac{1}{\alpha_0}+3\epsilon+4\beta\varepsilon+\gamma_0}.\end{aligned}$$

Now, since $(3+4\beta)\epsilon<\frac{1}{\alpha_0}$ we conclude that

$$\limsup_{n\to+\infty}\sum_{x,y,z\in\mathbb{Z}^{d_0}}a_n^{-2\beta}|\zeta_{n,x,z}|^{\beta}|\zeta_{n,x,y}|^{\beta}=0.\tag{59}$$

Observe moreover that, due to Item (iv) of Assumption 1, we have

$$\begin{aligned}
&\mathbb{E}\left[|U_{n,x,y}U_{n,x,z}||\tilde{\mathcal{F}}\right] \\
\leq\ & \int_{(0,\delta)^2} \mathbb{P}(a_n^{-1}|h(\xi_1,\xi_2)\zeta_{n,x,y}|>u, a_n^{-1}|h(\xi_1,\xi_3)\zeta_{n,x,z}|>v|\tilde{\mathcal{F}})\,dudv \\
\leq\ & C_0\left[a_n^{-1}|\zeta_{n,x,y}| + \int_{a_n^{-1}|\zeta_{n,x,z}|}^{\delta} u^{-\gamma}a_n^{-\gamma}|\zeta_{n,x,y}|^{\gamma}\,du\right] \\
& \times\left[a_n^{-1}|\zeta_{n,x,z}| + \int_{a_n^{-1}|\zeta_{n,x,z}|}^{\delta} v^{-\gamma}a_n^{-\gamma}|\zeta_{n,x,z}|^{\gamma}\,dv\right] \\
\leq\ & C_0\left[a_n^{-1}|\zeta_{n,x,y}|^1 + \frac{\delta^{1-\gamma}-a_n^{\gamma-1}|\zeta_{n,x,z}|^{1-\gamma}}{1-\gamma}a_n^{-\gamma}|\zeta_{n,x,y}|^{\gamma}\right] \\
& \times\left[a_n^{-1}|\zeta_{n,x,z}| + \frac{\delta^{1-\gamma}-a_n^{\gamma-1}|\zeta_{n,x,z}|^{1-\gamma}}{1-\gamma}a_n^{-\gamma}|\zeta_{n,x,z}|^{\gamma}\right] \\
\leq\ & C_\delta a_n^{-2\gamma'}|\zeta_{n,x,y}\zeta_{n,x,z}|^{\gamma'} \quad \text{where } \gamma'=\min(1,\gamma)
\end{aligned}$$

for n large enough and some $C_\delta>0$. Indeed, due to (37) we have $a_n^{-1}\sup_{x,y}|\zeta_{n,x,y}|\leq 1$ for large n. Again using (37) and to the definition of $\tilde{R}_n$, for every $\gamma_0>0$, we have

$$\begin{aligned}
\sum_{x,y,z\in\mathbb{Z}^{d_0}} \mathbb{E}\left[|U_{n,x,y}U_{n,x,z}||\tilde{\mathcal{F}}\right] &\leq C_\delta \tilde{R}_n^3 a_n^{-2\gamma'}\sup_{x,y}|\zeta_{n,x,y}|^{2\gamma'} \\
&\leq n^{\frac{3}{\alpha_0}-\frac{4\gamma'}{\alpha_0\beta}+7\varepsilon+\gamma_0},
\end{aligned}$$

for n large enough. Recall that we have chosen ε such that $\frac{3}{\alpha_0}-\frac{4\gamma'}{\alpha_0\beta}+7\varepsilon<0$. Hence, we obtain

$$\forall\delta>0,\quad \limsup_{n\to+\infty}\sum_{x,y,z}\mathbb{E}[|U_{n,x,y}U_{n,x,z}|]=0. \tag{60}$$

Now putting (56), (57), (58), (59) and (60) all together, we conclude that

$$\forall\delta>0,\ \limsup_{n\to+\infty}\sum_{(x,y,x',y')\in E_3}\mathbf{E}[T_{n,x,y}T_{n,x',y'}|\tilde{\mathcal{F}}]=0.$$

* Assume now that $\beta\geq\frac{4}{3}$. Observe that, with the notation of Item (vi) of Assumption 1, we have

$$T_{n,x,y}=a_n^{-1}\zeta_{n,x,y}\mathbf{h}_{(a_n\delta|\zeta_{n,x,y}|^{-1})}(\xi_x,\xi_y).$$

Due to this Item (vi), to the definition of $\tilde{R}_n$ and to (37), for every $\gamma_0>0$, we have almost surely

$$\begin{aligned}
\sum_{x,y,z\in\mathbb{Z}^{d_0}}|\mathbb{E}[T_{n,x,y}T_{n,x,z}|\tilde{\mathcal{F}}]| &\leq C_0'a_n^{-2}\sum_{x,y,z\in\mathbb{Z}^{d_0}}|\zeta_{n,x,y}\zeta_{n,x,z}|\left(a_n^2\delta^2|\zeta_{n,x,y}\zeta_{n,x',y'}|^{-1}\right)^{-\theta'} \\
&\leq \delta^{-2\theta'}\tilde{R}_n^3\left(a_n^{-1}(\tilde{N}_n^*)^2\right)^{2(\theta'+1)} \\
&\leq n^{\frac{1}{\alpha_0}\left(3-\frac{4(\theta'+1)}{\beta}\right)+(4\theta'+7)\varepsilon+\gamma_0},
\end{aligned}$$

for n large enough. Since $\frac{1}{\alpha_0}\left(3-\frac{4(\theta'+1)}{\beta}\right)+(4\theta'+7)\varepsilon<0$, we obtain

$$\forall\delta>0,\quad \limsup_{n\to+\infty}\sum_{(x,y,x',y')\in E_3}|\mathbb{E}[T_{n,x,y}T_{n,x,z}|\tilde{\mathcal{F}}]|=0.$$

So, finally, for $\beta\in[1,2)$, there exists $\tilde{C}>0$ such that, for every nonnegative n and every $\delta>0$, we have $\limsup_{n\to+\infty}\mathbf{E}[(T_n(\delta))^2]\le\tilde{C}\delta^{2-\beta}$.

□

Proof of Proposition 17. The following proof can be assembled from [13]. We will use the constants $I_0:=-\int_0^\infty\frac{\sin y}{y^\beta}\,dy$ and $J_0:=-\tan\frac{\pi\beta}{2}I_0$. Due to the exponential formula, we have

$$\begin{aligned}\mathbb{E}\left[e^{it\int_{\{|x|\ge\delta\}}x\,d\mathcal{P}(x)}\right] &= \exp\left(\int_{\{|x|\ge\delta\}}(e^{itx}-1)(a\mathbf{1}_{\{x>0\}}+b\mathbf{1}_{\{x<0\}})\beta|x|^{-\beta-1}\,dx\right)\\ &= \exp\left((a+b)\int_\delta^{+\infty}\frac{\cos(tx)-1}{x^{\beta+1}}\beta\,dx+i(a-b)\int_\delta^{+\infty}\frac{\sin(tx)}{x^{\beta+1}}\beta\,dx\right)\end{aligned}$$

Assume first that $\beta<1$. Due to [13, p. 568], we have

$$\lim_{\delta\to0}\int_\delta^{+\infty}\frac{e^{itx}-1}{x^{\beta+1}}\beta\,dx=-|t|^\beta\Gamma(1-\beta)e^{-\frac{i\pi\beta}{2}}=|t|^\beta(I_0+iJ_0).$$

So $\lim_{\delta\to0}\mathbb{E}\left[e^{it\int_{\{|x|\ge\delta\}}x\,d\mathcal{P}(x)}\right]=\Phi_{a+b,a-b,\beta}(t)$.
Assume now that $\beta=1$. Then

$$\lim_{\delta\to0}\int_\delta^{+\infty}\frac{\cos(tx)-1}{x^2}\,dx=\int_0^{+\infty}\frac{\cos(tx)-1}{x^2}\,dx=|t|\int_0^{+\infty}\frac{\cos(y)-1}{y^2}\,dy=-\frac{\pi}{2}|t|$$

and, since $\sin(tx)=\operatorname{sgn}(t)\sin(|t|x)$, we have

$$\int_\delta^{+\infty}\frac{\sin(tx)}{x^2}\,dx=t\int_{\delta|t|}^{+\infty}\frac{\sin y}{y^2}\,dy$$

and so

$$\int_\delta^{+\infty}\frac{\sin(tx)}{x^2}\,dx-t\int_\delta^{+\infty}\frac{\sin x}{x^2}\,dx=t\int_{\delta|t|}^{\delta}\frac{\sin y}{y^2}\,dy\sim_{\delta\to0}t\int_{\delta|t|}^{\delta}\frac{dy}{y}=-t\log|t|.$$

Hence we have in that case that

$$\lim_{\delta\to0}\mathbb{E}\left[\exp\left(it\left(\int_{|x|>\delta}xd\mathcal{P}(x)-(a-b)\int_\delta^\infty\frac{\sin x}{x^2}dx\right)\right)\right]=\Phi_{a+b,a-b,1}(t).$$

Assume finally $\beta>1$. Due to [13, p.568-569], we have

$$\lim_{\delta\to0}\int_\delta^\infty\frac{e^{itx}-1-itx}{x^{\beta+1}}\beta\,dx=\int_0^{+\infty}\frac{e^{itx}-1-itx}{x^{\beta+1}}\beta\,dx=|t|^\beta\frac{\Gamma(3-\beta)e^{-\frac{i\pi\beta}{2}}}{(2-\beta)(\beta-1)}=|t|^\beta(I_0+iJ_0).$$

So

$$\lim_{\delta\to0}\mathbb{E}\left[e^{it\int_{\{|x|\ge\delta\}}x\,d\mathcal{P}(x)-it(a-b)\beta\frac{\delta^{1-\beta}}{\beta-1}}\right]=\Phi_{a+b,a-b,\beta}(t).$$

□

5. Tightness

Here we treat case $\alpha_0 > 1$ (i.e. the case where $(S_n)_n$ is recurrent and $\alpha > d_0 = 1$). The tightness proof follows essentially the one given in Kesten and Spitzer [17]. We need the following lemma from [17].

Lemma 19 (Lemma 1 of [17])**.** *For all $\epsilon > 0$ there exists some $A > 0$ such that for all $t \geq 1$ one has*

$$\mathbb{P}\left(\exists x \in \mathbb{Z} : |x| > At^{1/\alpha} \text{ and } N_t(x) > 0\right) \leq \epsilon.$$

Lemma 20. *We have*

$$\mathbb{E}\left[\sum_{x\in\mathbb{Z}} N_n^2(x)\right] = O(n^{2-\frac{1}{\alpha}}) \quad \text{and} \quad \mathbb{E}\left[\Big(\sum_{x\in\mathbb{Z}} N_n^2(x)\Big)^2\right] = O(n^{4-\frac{2}{\alpha}}). \tag{61}$$

Proof. The first one is formula (2.13) from [17] and the second one can be found in [15, Lemma 2.1]. □

Proposition 21. *The sequence of stochastic processes*

$$U_t^n := n^{-2\delta} \sum_{x,y\in\mathbb{Z}} N_{\lfloor nt\rfloor}(x) N_{\lfloor nt\rfloor}(y) h(\xi_x, \xi_y); \;\; t \geq 0$$

is tight in $C(0,T)$ with sup-norm.

Proof. It is sufficient to show that

$$\limsup_{n\to\infty} \lim_{\kappa\downarrow 0} \sup_{0\leq t_1,t_2\leq T:|t_1-t_2|\leq\kappa} \mathbb{P}\left(\left|U_{t_1}^n - U_{t_2}^n\right| > \eta\right) = 0.$$

Fix some $\epsilon > 0$. Due to Lemma 19, we fix $A > 0$ large enough such that

$$\mathbb{P}\Big(\exists x \in \mathbb{Z} \text{ with } |x| > An^{1/\alpha} \text{ and } N_{\lfloor nT\rfloor}(x) > 0\Big) \leq \frac{\epsilon}{4}. \tag{62}$$

Choose some $\rho > 0$ such that for all $n \in \mathbb{N}$ one has

$$9A^2 n^{2/\alpha} \mathbb{P}\left(|h(\xi_1, \xi_2)| > \rho n^{\frac{2}{\alpha\beta}}\right) < \frac{\epsilon}{4}. \tag{63}$$

This is possible since we have, by Item (iii) of Assumption 1, that

$$\lim_{u\to\infty} u^\beta \mathbb{P}(h(\xi_1,\xi_2) \geq u) = c_0 \quad \text{and} \quad \lim_{u\to\infty} u^\beta \mathbb{P}(h(\xi_1,\xi_2) \leq -u) = c_1. \tag{64}$$

Define

$$\bar{h}(x,y) := h(x,y)\mathbf{1}_{\{|h(x,y)|\leq\rho n^{\frac{2}{\alpha\beta}}\}}.$$

The inequality (63) now becomes

$$9A^2 n^{2/\alpha} \mathbb{P}\left(\bar{h}(\xi_1,\xi_2) \neq h(\xi_1,\xi_2)\right) \leq \frac{\epsilon}{4}. \tag{65}$$

Lemma 22. *There exists a constant $C = C(\rho, \beta) > 0$ such that for all $n \geq 1$ one has*

$$\left|\mathbb{E}\left[\bar{h}(\xi_1,\xi_2)\right]\right| \leq C n^{(1-\beta)\frac{2}{\alpha\beta}}. \tag{66}$$

Proof. For $\beta < 1$, we have

$$\begin{aligned} \left|\mathbb{E}\left[\bar{h}(\xi_1,\xi_2)\right]\right| &\leq \int_0^{\rho n^{\frac{2}{\alpha\beta}}} \mathbb{P}\left(|h(\xi_1,\xi_2)| > x\right) dx \leq C \int_1^{\rho n^{\frac{2}{\alpha\beta}}} x^{-\beta} dx + 1 \\ &= C x^{1-\beta}\Big|_1^{\rho n^{2/\alpha\beta}} + 1 \sim C n^{\frac{2}{\alpha\beta}(1-\beta)} \end{aligned}$$

where $C > 0$ is some suitable constant. For $\beta \in (1,2)$, this comes from (52). For $\beta = 1$, as noticed previously, this comes from Item (vii) of Assumption 1. □

Now we define

$$E_n := n^{-2\delta}\mathbb{E}\left[\sum_{x,y\in\mathbb{Z}} N_n(x)N_n(y)\bar{h}(\xi_x,\xi_y)\right].$$

Since the scenery and the random walk are independent, we compute

$$\begin{aligned} E_n &= n^{-2\delta}\mathbb{E}\left[\sum_{x,y\in\mathbb{Z}} N_n(x)N_n(y)\mathbb{E}\left[\bar{h}(\xi_x,\xi_y)\right]\right] = n^{-2\delta}n^2\mathbb{E}\left[\bar{h}(\xi_1,\xi_2)\right] \\ &\leq Cn^{-2+\frac{2}{\alpha}-\frac{2}{\alpha\beta}}n^2 n^{(1-\beta)\frac{2}{\alpha\beta}} = C, \end{aligned}$$

due to Lemma 22. Thus the sequence E_n stays bounded as $n\to\infty$. Further, let

$$\bar{U}_t^n := n^{-2\delta}\sum_{x,y\in\mathbb{Z}} N_{\lfloor nt\rfloor}(x)N_{\lfloor nt\rfloor}(y)\left(\bar{h}(\xi_x,\xi_y)-\mathbb{E}\left[\bar{h}(\xi_x,\xi_y)\right]\right).$$

It then follows

$$\begin{aligned} U_t^n-\bar{U}_t^n-t^2E_n &= n^{-2\delta}\sum_{x,y\in\mathbb{Z}} N_{\lfloor nt\rfloor}(x)N_{\lfloor nt\rfloor}(y)\left(h(\xi_x,\xi_y)-\bar{h}(\xi_x,\xi_y)\right) \\ &\quad + n^{-2\delta}\Big(\lfloor nt\rfloor^2\mathbb{E}\left[\bar{h}(\xi_1,\xi_2)\right]-n^2t^2\mathbb{E}\left[\bar{h}(\xi_1,\xi_2)\right]\Big). \end{aligned}$$

Since we have that $\mathbb{E}[\bar{h}(\xi_1,\xi_2)] = O(n^{(1-\beta)\frac{2}{\alpha\beta}})$ and $\lfloor nt\rfloor^2-n^2t^2 = O(n)$ the second term is of the order

$$n^{-2\delta}O(n^{(1-\beta)\frac{2}{\alpha\beta}})(\lfloor nt\rfloor^2-n^2t^2) = n^{-2}O(n) = O(n^{-1}).$$

This implies with inequalities (62) and (65) that

$$\begin{aligned} &\limsup_{n\to\infty}\mathbb{P}\left(\sup_{0\leq t\leq T}\left|U_t^n-\bar{U}_t^n-t^2E_n\right| > \frac{\eta}{2}\right) \\ \leq\ &\limsup_{n\to\infty}\mathbb{P}\left(n^{-2\delta}\sum_{x,y\in\mathbb{Z}} N_{\lfloor nT\rfloor}(x)N_{\lfloor nT\rfloor}(y)\left(h(\xi_x,\xi_y)-\bar{h}(\xi_x,\xi_y)\right) > \frac{\eta}{4}\right) \\ \leq\ &\limsup_{n\to\infty}\mathbb{P}\left(\sum_{x,y\in\mathbb{Z}} N_{\lfloor nT\rfloor}(x)N_{\lfloor nT\rfloor}(y)\left(h(\xi_x,\xi_y)-\bar{h}(\xi_x,\xi_y)\right) \neq 0\right) \\ \leq\ &\limsup_{n\to\infty}\mathbb{P}\left(\exists x,y\in\mathbb{Z}:|x|,|y|\leq An^{1/\alpha},\bar{h}(\xi_x,\xi_y)\neq h(\xi_x,\xi_y)\right) \\ &+\limsup_{n\to\infty}\mathbb{P}\left(\exists x\in\mathbb{Z}:\ |x|>An^{1/\alpha},N_{\lfloor nT\rfloor}(x)>0\right) \\ \leq\ &\limsup_{n\to\infty}\ (3An^{1/\alpha})^2\mathbb{P}\left(\bar{h}(\xi_1,\xi_2)\neq h(\xi_1,\xi_2)\right)+\ \frac{\epsilon}{4} \\ \leq\ &\frac{\epsilon}{2}. \end{aligned}$$

It is now sufficient to prove that

$$\limsup_{n\to\infty}\lim_{\kappa\downarrow 0}\sup_{0\leq t_1,t_2\leq T:|t_1-t_2|\leq\kappa}\mathbb{P}\left(\left|\bar{U}_{t_1}^n-\bar{U}_{t_2}^n\right|>\frac{\eta}{2}\right)=0.$$

For this we prove for all $T\geq t>s\geq 0$ that

$$\mathbb{E}\left[\left(\bar{U}_t^n-\bar{U}_s^n\right)^2\right] \leq C(t-s)^{2-\frac{2}{\alpha}}. \tag{67}$$

If we use the notation

$$\bar{h}_0(\xi_x,\xi_y) := \bar{h}(\xi_x,\xi_y) - \mathbb{E}\left[\bar{h}(\xi_x,\xi_y)\right]$$

then we have

$$\begin{aligned}
\mathbb{E}\left[\left(\bar{U}_t^n - \bar{U}_s^n\right)^2\right] &= n^{-4\delta}\mathbb{E}\Bigg[\Bigg(\sum_{x,y} N_{\lfloor nt\rfloor}(x)\Big(N_{\lfloor nt\rfloor}(y) - N_{\lfloor ns\rfloor}(y)\Big)\bar{h}_0(\xi_x,\xi_y) \\
&\qquad + \sum_{x,y}\Big(N_{\lfloor nt\rfloor}(x) - N_{\lfloor ns\rfloor}(x)\Big)N_{\lfloor ns\rfloor}(y)\bar{h}_0(\xi_x,\xi_y)\Bigg)^2\Bigg] \\
&\leq 2n^{-4\delta}\mathbb{E}\Bigg[\Bigg(\sum_{x,y} N_{\lfloor nt\rfloor}(x)\Big(N_{\lfloor nt\rfloor}(y) - N_{\lfloor ns\rfloor}(y)\Big)\bar{h}_0(\xi_x,\xi_y)\Bigg)^2\Bigg] \\
&\qquad + 2n^{-4\delta}\mathbb{E}\Bigg[\Bigg(\sum_{x,y}\Big(N_{\lfloor nt\rfloor}(x) - N_{\lfloor ns\rfloor}(x)\Big)N_{\lfloor ns\rfloor}(y)\bar{h}_0(\xi_x,\xi_y)\Bigg)^2\Bigg].
\end{aligned}$$

We continue the computation with the first of the two terms. In the following we condition with respect to $\mathcal{G} = \sigma(S_n; n \in \mathbb{N})$. We make use of the assumption $h(x,x) = 0$ and the fact that if x,y,u,v are all distinct then $\bar{h}_0(\xi_x,\xi_y)$ and $\bar{h}_0(\xi_u,\xi_v)$ are independent and centered and we write

$$\mathbb{E}\Bigg[\Bigg(\sum_{x,y} N_{\lfloor nt\rfloor}(x)\Big(N_{\lfloor nt\rfloor}(y) - N_{\lfloor ns\rfloor}(y)\Big)\bar{h}_0(\xi_x,\xi_y)\Bigg)^2\Bigg|\mathcal{G}\Bigg] \leq A+B+C+D$$

with

$$\begin{aligned}
A &:= \sum_{x,y} N_{\lfloor nt\rfloor}^2(x)\Big(N_{\lfloor nt\rfloor}(y) - N_{\lfloor ns\rfloor}(y)\Big)^2\mathbb{E}\left[\bar{h}_0^2(\xi_1,\xi_2)\middle|\mathcal{G}\right], \\
B &:= \sum_{x,y,z} N_{\lfloor nt\rfloor}(x)N_{\lfloor nt\rfloor}(z)\Big(N_{\lfloor nt\rfloor}(y) - N_{\lfloor ns\rfloor}(y)\Big)^2\mathbb{E}\left[|\bar{h}_0(\xi_1,\xi_2)\bar{h}_0(\xi_2,\xi_3)|\middle|\mathcal{G}\right],
\end{aligned}$$

$$C := \sum_{x,y,z} N_{\lfloor nt\rfloor}^2(x)\Big(N_{\lfloor nt\rfloor}(y) - N_{\lfloor ns\rfloor}(y)\Big)\Big(N_{\lfloor nt\rfloor}(z) - N_{\lfloor ns\rfloor}(z)\Big)\mathbb{E}\left[|\bar{h}_0(\xi_1,\xi_2)\bar{h}_0(\xi_2,\xi_3)|\middle|\mathcal{G}\right].$$

and

$$D := 2\sum_{x,x',y} N_{\lfloor nt\rfloor}(x')N_{\lfloor nt\rfloor}(x)\Big(N_{\lfloor nt\rfloor}(y) - N_{\lfloor ns\rfloor}(y)\Big)\Big(N_{\lfloor nt\rfloor}(x) - N_{\lfloor ns\rfloor}(x)\Big)\mathbb{E}\left[|\bar{h}_0(\xi_1,\xi_2)\bar{h}_0(\xi_2,\xi_3)|\middle|\mathcal{G}\right].$$

The Markov property together with Lemma 20 and Lemma 23 below imply

$$\begin{aligned}
\mathbb{E}[B] &\leq T^2n^2\mathbb{E}\left[\sum_x N_{\lfloor nt\rfloor - \lfloor ns\rfloor}^2(x)\right]\mathrm{Cov}\left(\bar{h}(\xi_1,\xi_2),\bar{h}(\xi_2,\xi_3)\right) \\
&\leq C'n^2n^{2-\frac{1}{\alpha}}(t-s)^{2-\frac{1}{\alpha}}n^{-\frac{3}{\alpha}+\frac{4}{\alpha\beta}} \\
&= (t-s)^{2-\frac{1}{\alpha}}O(n^{4\delta}).
\end{aligned}$$

Again we see

$$\begin{aligned}
\mathbb{E}[C] &= (\lfloor nt\rfloor - \lfloor ns\rfloor)^2\mathbb{E}\left[\sum_x N_{\lfloor nt\rfloor}^2(x)\right]\mathrm{Cov}\left(\bar{h}(\xi_1,\xi_2),\bar{h}(\xi_2,\xi_3)\right) \\
&\leq n^2(t-s)^2n^{2-\frac{1}{\alpha}}T^{2-\frac{1}{\alpha}}n^{-\frac{3}{\alpha}+\frac{4}{\alpha\beta}} \\
&= (t-s)^2O(n^{4\delta}).
\end{aligned}$$

Further, we have by Cauchy-Schwarz that

$$\begin{aligned}\mathbb{E}\left[\sum_x N_{\lfloor nt\rfloor}(x)\Big(N_{\lfloor nt\rfloor}(x)-N_{\lfloor ns\rfloor}(x)\Big)\right] &\le \left(\mathbb{E}\left[\sum_x N^2_{\lfloor nt\rfloor}(x)\right]\mathbb{E}\left[\sum_x\Big(N_{\lfloor nt\rfloor}(x)-N_{\lfloor ns\rfloor}(x)\Big)^2\right]\right)^{\frac12}\\ &\le C'(nt)^{1-\frac{1}{2\alpha}}(n(t-s))^{1-\frac{1}{2\alpha}}.\end{aligned}$$

Now Lemma 23 implies

$$\begin{aligned}\mathbb{E}\left[D\right] &\le [nt]([nt)-[ns])\mathbb{E}\left[\sum_x N_{\lfloor nt\rfloor}(x)\Big(N_{\lfloor nt\rfloor}(x)-N_{\lfloor ns\rfloor}(x)\Big)\right]\operatorname{Cov}\left(\bar h(\xi_1,\xi_2),\bar h(\xi_2,\xi_3)\right)\\ &\le C''n(n(t-s))(nt)^{1-\frac{1}{2\alpha}}(n(t-s))^{1-\frac{1}{2\alpha}}n^{-\frac{3}{\alpha}+\frac{4}{\alpha\beta}}\end{aligned}$$

For $t-s<\kappa<1$ this is smaller than $C''(t-s)^{2-\frac{2}{\alpha}}O(n^{4\delta})$. Finally for A, due to Lemma 24 below, we have

$$\begin{aligned}\mathbb{E}\left[A\right] &\le \sqrt{\mathbb{E}\left[\left(\sum_x N^2_{nt}(x)\right)^2\right]\mathbb{E}\left[\left(\sum_y N^2_{n(t-s)}(y)\right)^2\right]}\operatorname{Var}\left(\bar h(\xi_1,\xi_2)\right)\\ &\le C''\sqrt{O((tn)^{4-\frac{2}{\alpha}})O(((t-s)n)^{4-\frac{2}{\alpha}}})\mathbb{E}[(\bar h(\xi_1,\xi_2))^2]\\ &\le C'''(t-s)^{2-\frac{1}{\alpha}}n^{4-\frac{2}{\alpha}}n^{-\frac{2}{\alpha}+\frac{4}{\alpha\beta}}\\ &\le C'''(t-s)^{2-\frac{1}{\alpha}}n^{4\delta}.\end{aligned}$$

All those inequalities together prove that there exists some constant $K>0$ such that for $(t-s)<\kappa<1$ one has

$$\mathbb{E}\left[\left(\bar U^n_t-\bar U^n_s\right)^2\right]\le K(t-s)^{2-\frac{2}{\alpha}}.$$

This finishes the tightness proof. □

Lemma 23. *There is some constant $C>0$ such that*

$$\left|\operatorname{Cov}\left(\bar h(\xi_1,\xi_2),\bar h(\xi_1,\xi_3)\right)\right|\le C'n^{-\frac{3}{\alpha}+\frac{4}{\alpha\beta}}.$$

Proof. We first do the case $\beta < \frac{4}{3}$. Note that by Assumption 1 part (iv) for some $\gamma > \frac{3\beta}{4}$ ($\gamma \neq 1$), we have

$$\begin{aligned}
\mathbb{E}\left[|\bar{h}(\xi_1,\xi_2)\bar{h}(\xi_1,\xi_3)|\right] &= \int_0^\infty\int_0^\infty \mathbb{P}\left(|\bar{h}(\xi_1,\xi_2)|>s,|\bar{h}(\xi_1,\xi_3)|>t\right)dsdt \\
&= \int_0^{\rho n^{\frac{2}{\alpha\beta}}}\int_0^{\rho n^{\frac{2}{\alpha\beta}}} \mathbb{P}\left(|h(\xi_1,\xi_2)|>s,|h(\xi_1,\xi_3)|>t\right)dsdt \\
&\leq \int_0^{\rho n^{\frac{2}{\alpha\beta}}}\int_0^{\rho n^{\frac{2}{\alpha\beta}}} C_0(\max(1,s)\max(1,t))^{-\gamma}dsdt \\
&= C_0\left(1+\int_1^{\rho n^{\frac{2}{\alpha\beta}}} t^{-\gamma}dt\right)^2 \\
&\leq C_0\left(1+\frac{1}{1-\gamma}\left(\left(\rho n^{\frac{2}{\alpha\beta}}\right)^{1-\gamma}-1\right)\right)^2 \\
&\leq C_0\left(1-\frac{1}{1-\gamma}+\frac{\rho^{1-\gamma}}{1-\gamma}n^{-\frac{3}{2\alpha}+\frac{2}{\alpha\beta}}\right)^2 = O(n^{-\frac{3}{\alpha}+\frac{4}{\alpha\beta}})
\end{aligned}$$

Due to Lemma 22 this implies

$$\left|\mathrm{Cov}\left(\bar{h}(\xi_1,\xi_2),\bar{h}(\xi_1,\xi_3)\right)\right| = O(n^{-\frac{3}{\alpha}+\frac{4}{\alpha\beta}}).$$

Now assume $\beta \geq \frac{4}{3}$. By (52) and Item (vi) of Assumption 1, we have for $M_n := \rho n^{\frac{2}{\alpha\beta}}$ that

$$\begin{aligned}
\left|\mathrm{Cov}\left(\bar{h}(\xi_1,\xi_2),\bar{h}(\xi_1,\xi_3)\right)\right| &= \left|\mathrm{Cov}\left(\mathbf{h}_{M_n}(\xi_1,\xi_2),\mathbf{h}_{M_n}(\xi_1,\xi_3)\right)\right| \\
&\leq \left|\mathbb{E}\left[\mathbf{h}_{M_n}(\xi_1,\xi_2)\mathbf{h}_{M_n}(\xi_1,\xi_3)\right]\right| + |\mathbb{E}[\mathbf{h}_{M_n}(\xi_1,\xi_2)]|^2 \\
&\leq O\left(n^{-\frac{4\theta'}{\alpha\beta}}\right) + O\left(n^{-\frac{4}{\alpha\beta}(\beta-1)}\right) \\
&\leq O\left(n^{-\frac{4}{\alpha\beta}\left(\frac{3\beta}{4}-1\right)}\right) \\
&= O(n^{-\frac{3}{\alpha}+\frac{4}{\alpha\beta}})
\end{aligned}$$

since $\theta' > \frac{3\beta}{4} - 1$. □

Lemma 24. *We have*

$$\mathbb{E}\left[\bar{h}(\xi_1,\xi_2))^2\right] = O\left(n^{-\frac{2}{\alpha}+\frac{4}{\alpha\beta}}\right).$$

Proof. We have

$$\begin{aligned}
\mathbb{E}\left[\bar{h}(\xi_1,\xi_2))^2\right] &= \int_0^{\rho n^{\frac{2}{\alpha\beta}}} \mathbb{P}(|\bar{h}(\xi_1,\xi_2)|^2 \geq s)\,ds = \int_0^{\rho n^{\frac{2}{\alpha\beta}}} \mathbb{P}(|\bar{h}(\xi_1,\xi_2)| \geq u)2u\,du \\
&= O(n^{\frac{2}{\alpha\beta}(2-\beta)}),
\end{aligned}$$

since $2u\mathbb{P}(|\bar{h}(\xi_1,\xi_2)| \geq u) \sim 2(c_0+c_1)u^{1-\beta}$ as u goes to infinity. □

Appendix A. Stochastic integral with respect to the Lévy sheet Z

In this section, following [18], we give a simple construction of stochastic integral with respect to the β-stable Lévy sheet Z. In [18], Khoshnevisan and Nualart considered general Lévy sheet with symmetric distributions. Therefore their results apply to the β-stable Lévy sheet Z only if $c_0 = c_1$. Nevertheless, we will see that their construction is expansible when $c_0 \neq c_1$.

Let us recall that Z satisfies the following properties:

- $Z_{0,0} = 0$;
- for any family $(A_k = [a_k, b_k] \times [a'_k, b'_k])_k$ of pairwise disjoint rectangles (with $a_k < b_k$ and $a'_k < b'_k$), the family of increments $(Z_{b_k,b'_k} + Z_{a_k,a'_k} - Z_{a_k,b'_k} - Z_{b_k,a'_k})_k$ is a family of independent random variables;
- for any rectangle $A = [a, b] \times [a', b']$ (with $a < b$ and $a' < b'$), the characteristic function of the increment $Z_{b,b'} + Z_{a,a'} - Z_{a,b'} - Z_{b,a'}$ is $\Phi_{(c_0+c_1)\lambda(A),(c_0-c_1)\lambda(A),\beta}$, where λ is the Lebesgue measure on $\mathbb{R}^2$ and where we used the notation introduced in (4).

For any rectangle $A = [a, b] \times [a', b']$ (with $a < b$ and $a' < b'$), we define the stochastic integral of $\mathbf{1}_A$ with respect to the Lévy process as the increment of Z in this rectangle, i.e.

$$\int_{\mathbb{R}^2} \mathbf{1}_A \, dZ_{x,y} := Z_{b,b'} + Z_{a,a'} - Z_{a,b'} - Z_{b,a'}. \tag{68}$$

We extend this definition by linearity to any linear combination H of such indicator functions. Observe that, if $H = \sum_{j=1}^{\mu} h_j \mathbf{1}_{A_j}$ where $(A_j)_j$ is a family of pairwise disjoint rectangles and where $h_j \in \mathbb{R}$, then the characteristic function of $\int_{\mathbb{R}^2} H(x, y)\, dZ_{x,y}$ is given by

$$\begin{aligned}
\forall z \in \mathbb{R}, \quad \mathbb{E}\left[\exp\left(iz \int_{\mathbb{R}^2} H(x,y)\, dZ_{x,y}\right)\right] &= \prod_{j=1}^{\mu} \mathbb{E}\left[\exp\left(izh_j \int_{\mathbb{R}^2} \mathbf{1}_{A_j}(x,y)\, dZ_{x,y}\right)\right] \\
&= \prod_{j=1}^{\mu} \Phi_{(c_0+c_1)\lambda(A_j),(c_0-c_1)\lambda(A_j),\beta}(zh_j) \\
&= \prod_{j=1}^{\mu} \Phi_{(c_0+c_1)|h_j|_+^\beta \lambda(A_j),(c_0-c_1)|h_j|_-^\beta \lambda(A_j),\beta}(z) \\
&= \Phi_{(c_0+c_1)\sum_{j=1}^{\mu} |h_j|_+^\beta \lambda(A_j),(c_0-c_1)\sum_{j=1}^{\mu} |h_j|_-^\beta \lambda(A_j),\beta}(z)
\end{aligned}$$

and so by

$$\forall z \in \mathbb{R}, \quad \mathbb{E}\left[\exp\left(iz \int_{\mathbb{R}^2} H(x,y)\, dZ_{x,y}\right)\right] = \Phi_{(c_0+c_1)\int_{\mathbb{R}^2} |H(x,y)|_+^\beta \, dxdy,(c_0-c_1)\int_{\mathbb{R}^2} |H(x,y)|_-^\beta \, dxdy,\beta}(z)). \tag{69}$$

Proposition 25. *(see* [18]*) Let H be a continuous compactly supported function from $\mathbb{R}^2$ to $\mathbb{R}$. Let $(H_n)_n$ be a sequence of linear combination of indicators over rectangles converging pointwise to H. Assume moreover that $(H_n)_n$ is a family of uniformly bounded functions with support in a same compact. Then the sequence $\left(\int_{\mathbb{R}^2} H_n(x,y)\, dZ(x,y)\right)_n$ converges in probability to a random variable with characteristic function $\Phi_{(c_0+c_1)\int_{\mathbb{R}^2} |H(x,y)|_+^\beta \, dxdy,(c_0-c_1)\int_{\mathbb{R}^2} |H(x,y)|_-^\beta \, dxdy,\beta}$.*

For a continuous compactly supported $H : \mathbb{R}^2 \to \mathbb{R}$, we define $\int_{\mathbb{R}^2} H(x,y)\, dZ(x,y)$ as the limit in probability given by Proposition 25 (observe that the limit does not depend on the choice of $(H_n)_n$).

Proof of Proposition 25. To prove the convergence in probability, it is enough to prove that

$$\forall z \in \mathbb{R}, \quad \lim_{n,m\to+\infty} \mathbb{E}\left[\exp\left(iz \int_{\mathbb{R}} (H_n(x,y) - H_m(x,y))\, dZ_{x,y}\right)\right] = 1. \tag{70}$$

Observe that, for every real number z, we have

$$\begin{aligned}
&\left|\mathbb{E}\left[\exp\left(iz \int_{\mathbb{R}^2} (H_n(x,y) - H_m(x,y))\, dZ_{x,y}\right)\right] - 1\right| \\
= &\left|\Phi_{(c_0+c_1)\int_{\mathbb{R}^2} |H_n(x,y)-H_m(x,y)|_+^\beta \, dxdy,(c_0-c_1)\int_{\mathbb{R}^2} |H_n(x,y)-H_m(x,y)|_+^\beta \, dxdy,\beta}(z) - 1\right| \\
\leq &\ C \int_{\mathbb{R}^2} |H_n(x,y) - H_m(x,y)|^\beta \, dxdy(|c_0+c_1| + |c_0-c_1|)|z|^\beta,
\end{aligned}$$

using the fact that $|e^{-a+ib} - e^{-a'+ib'}| \le |a-a'| + |b-b'|$ for any real numbers a, b, a', b' such that $a > 0$ and $a' > 0$. Since $(H_n)_n$ converges pointwise and is uniformly bounded, we obtain (70) by the Lebesgue dominated convergence theorem (recall that $(H_n)_n$ is a sequence of uniformly bounded functions supported in a same compact). Now the characteristic function of the limit in probability $\int_{\mathbb{R}^2} H(x,y)\,dZ(x,y)$ is given by

$$\begin{aligned}\mathbb{E}\left[\exp\left(iz\int_{\mathbb{R}^2} H(x,y)\,dZ(x,y)\right)\right] &= \lim_{n\to+\infty} \mathbb{E}\left[\exp\left(iz\int_{\mathbb{R}^2} H_n(x,y)\,dZ(x,y)\right)\right] \\ &= \lim_{n\to+\infty} \Phi_{(c_0+c_1)\int_{\mathbb{R}^2}|H_n(x,y)|_+^\beta\,dxdy,(c_0-c_1)\int_{\mathbb{R}^2}|H_n(x,y)|_-^\beta\,dxdy,\beta}(z)) \\ &= \Phi_{(c_0+c_1)\int_{\mathbb{R}^2}|H(x,y)|_+^\beta\,dxdy,(c_0-c_1)\int_{\mathbb{R}^2}|H(x,y)|_-^\beta\,dxdy,\beta}(z)),\end{aligned}$$

for every real number z. □

References

[1] A.D. Barbour, G.K. Eagleson, Poisson convergence for dissociated statistics. *Journal of the Royal Statistical Society Ser. B* **46** (1984) 397-402.

[2] Billingsley, *Convergence of probability measure*. Second edition. John Wiley & Sons, Inc., New York, (1999).

[3] E. Bolthausen, A central limit theorem for two-dimensional random walks in random sceneries. *Annals of Probability* **17** (1989) 108-115.

[4] A. N. Borodin, A limit theorem for sums of independent random variables defined on a recurrent random walk. *Dokl. Akad. Nauk SSSR* **246** (1979) 786–787 (in Russian).

[5] A. N. Borodin, Limit theorems for sums of independent random variables defined on a transient random walk. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **85** (1979), 17-29, 237, 244 (in Russian).

[6] P. Cabus, N. Guillotin-Plantard, Functional limit theorems for U-statistics indexed by a random walk. *Stochastic Processes and their Applications* **101** (2002) 143-160.

[7] F. Castell, N. Guillotin-Plantard, F. Pène, Limit theorems for one and two-dimensional random walks in random scenery. *Ann. Inst. H. Poincaré* **49** (2013) 506-528.

[8] F. Castell, N. Guillotin-Plantard, F. Pène, Br. Schapira, A local limit theorem for random walks in random scenery and on randomly oriented lattices. *Annals of Probability* **39** (2011) 2079-2118.

[9] J. Cerný, Moments and distribution of the local time of a two-dimensional random walk. *Stochastic Processes and their Applications* **117** (2007) 262-270.

[10] A. Dabrowski, H. Dehling, T. Mikosch, O.Sh. Sharipov, Poisson limits for U-statistics, *Stochastic Processes and their Applications* **99** (2002) 137-157.

[11] G. Deligiannidis, S. Utev, An asymptotic variance of the self-intersections of random walks, *Sib. Math. J.* **52** (2011) 639–650.

[12] R.M. Dudley, Distances of probability measures and random variables, *Annals of Mathematical Statistics* **39** (1968) 1563-1572.

[13] W. Feller, An introduction to probability theory and its applications. Vol. II. Second edition, John Wiley & Sons, Inc., New-York, (1971).

[14] B. Franke, F. Pène, M. Wendler, Stable limit theorem for U-Statistic processes indexed by a random walk, arXiv:1212.2133.

[15] N. Guillotin-Plantard, V. Ladret, Limit theorems for U-statistics indexed by a one dimensional random walk, *ESAIM* **9** (2005) 95-115.

[16] N. C. Jain, W. E. Pruitt Asymptotic behavior of the local time of a recurrent random walk. *Annals of Probability* **12** (1984) 64-85.

[17] H. Kesten, F. Spitzer, A limit theorem related to an new class of self similar processes, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **50** (1979) 5-25.

[18] D. Khoshnevisan, E. Nualart, Level sets of the stochastic wave equation driven by a symmetric Lévy noise, *Bernoulli* **14** (2008) 899–925.

[19] J.F. Le Gall, J. Rosen, The range of stable random walks. *Annals of Probability* **19** (1991) 650–705.

[20] J. Neveu, Processus ponctuels, *Ecole d'été de Probabilité de Saint-Flour*, Lecture Notes in Mathematics, **598**, Springer Verlag, Berlin, (1976).

[21] S. I. Resnick, *Extreme values, regular variation, and point processes*. Springer Verlag, New York (1987).

[22] G. Samorodnitsky, M.S. Taqqu M.S., *Stable Non-Gaussian Random Processes*. Chapman & Hall., New York, (1994).

[23] F. Spitzer, *Principles of Random Walks*. Van Nostrand, Princeton, NJ (1964).

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